This week’s problems will continue with partial waves and also deal with resonant scattering and electron-atom scattering.

1. Consider a “shell potential,”

$$V(r) = \alpha \delta(r - r_0),$$

for \( r \), the radial spherical coordinate in three dimensions.

a. Find the \( s \)-state wavefunction for \( E > 0 \). Include an expression that determines the phase shift \( \delta_0 \). With \( \hbar k = \sqrt{2mE} \), show that in the limit \( k \to 0 \), \( \delta_0 \to ak \), with constant \( a \) (the “scattering length”). Solve for \( a \) in terms of \( \alpha \) and \( r_0 \).

b. How many bound states can exist for \( \ell = 0 \) and how does their existence depend on \( \alpha \)?

c. What is the scattering length \( a \) when a bound state appears at \( E = 0 \)? What happens to \( a \) as the shell potential changes from repulsive \((\alpha > 0)\) to attractive \((\alpha < 0)\), and when \( \alpha \) is sufficiently negative to form a bound state? Sketch \( a \) as a function of \( \alpha \).

2. In this problem you will consider the deuteron and neutron-proton scattering more carefully. The deuteron, a spin-triplet and \( \ell = 0 \) bound state of a neutron and a proton, has a binding energy of 2.26 MeV. It is also known that this is the only bound state of the neutron and proton (there are no excited bound states.) The neutron-proton scattering length (for the triplet) is \( a_t = 5.42 \text{ fm} \). Suppose we guess that the neutron-proton interaction is an attractive rectangular well of depth \( V_0 \) and radius \( R \) (i.e., \( V(r) = -V_0 \) for \( r < R \) and \( V(r) = 0 \) for \( r > R \).)

a. Find values for \( V_0 \) and \( R \) that reproduce this binding energy and scattering length.

b. Calculate the effective range \( r_0 \) you would expect from these parameters. How does this result compare with the experimental value of \( r_t = 1.73 \text{ fm} \)?

c. What is the total cross section at low energies? and how does it compare with \( \pi R^2 \)?

d. At what energies is it safe to approximate the total cross section just by the \( \ell = 0 \) cross section?
e. What is the total cross section for this potential at high energies? and how does it compare with \( \pi R^2 \)?

3. a. Derive equation (7.12.6) from Sakurai,

\[
\frac{d\sigma}{d\Omega} (0 \rightarrow n) = \frac{1}{(\hbar k/mL^3)} \frac{2\pi}{\hbar} |\langle k'n|V|0\rangle|^2 \left( \frac{L}{2\pi} \right)^3 \frac{(k'm)}{\hbar^2} \\
= \left( \frac{k'}{k} \right) L^6 \left[ \frac{1}{4\pi} \frac{2m}{\hbar^2} \langle k', n|V|k, 0\rangle \right]^2,
\]

the differential cross section for inelastic electron-atom scattering. Here, \( k \) is the wavevector of the incident particle (of mass \( m \)) and \( k' \) that for the scattered particle. The operator \( V \) describes the interaction potential between the incident electron and the atomic nucleus and electrons, and \( L \) is the size the (very large) box that is assumed to enclose the system.

b. (Sakurai 7.11) Show that the differential cross section for the elastic scattering of a fast electron by the ground state of the hydrogen atom is given by

\[
\frac{d\sigma}{d\Omega} = \left( \frac{4m^2e^4}{\hbar^4q^4} \right) \left\{ 1 - \frac{16}{[4 + (qa_0)^2]^2} \right\}^2.
\]

Ignore the effect of identity of the incident and atom electron.