This week’s problems will focus on the method of partial waves.

1. **Square-well potential** (Sakurai 7.3): Consider a potential $V = 0$ for $r > R$ and $V = V_0 = \text{constant}$ for $r < R$, where $V_0$ may be positive or negative. Using the method of partial waves, show that for $|V_0| \ll E = \hbar^2 k^2 / 2m$ and $kR \ll 1$ the differential cross section is isotropic and that the total cross section is given by

$$
\sigma_{\text{tot}} = \left( \frac{16\pi}{9} \right) \frac{m^2 V_0^2 R^6}{\hbar^4}.
$$

Suppose the energy is raised slightly. Show that that angular distribution can then be written as

$$
\frac{d\sigma}{d\Omega} = A + B \cos \theta,
$$

and obtain an approximate expression for $B/A$.

2. **Partial-wave scattering from a Yukawa potential** (Sakurai 7.4): A spinless particle is scattered by a weak Yukawa potential,

$$
V = \frac{V_0 e^{-\mu r}}{\mu r},
$$

where $\mu > 0$ but $V_0$ can be positive or negative. The first-order Born amplitude is given by

$$
f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2 \mu} \frac{1}{[2k^2(1 - \cos \theta) + \mu^2]}.
$$

a. Using $f^{(1)}(\theta)$ and assuming $|\delta_l| \ll 1$, obtain an expression for $\delta_l$ in terms of the Legendre function of the second kind,

$$
Q_l(\zeta) = \frac{1}{2} \int_{-1}^{1} \frac{P_l(\zeta')}{\zeta - \zeta'} d\zeta'.
$$
b. Use the expansion formula,

\[ Q_l(\zeta) = \frac{l!}{1 \cdot 3 \cdot 5 \cdots (2l + 1)} \]

\[ \times \left\{ \frac{1}{\zeta^{l+1}} + \frac{(l + 1)(l + 2)}{2(2l + 3)} \frac{1}{\zeta^{l+3}} + \frac{(l + 1)(l + 2)(l + 3)(l + 4)}{2 \cdot 4 \cdot (2l + 3)(2l + 5)} \frac{1}{\zeta^{l+5}} + \cdots \right\}, \quad (|\zeta| > 1) \]

to prove each of the following two assertions: (i) The phase shift \( \delta_l \) is negative (positive) when the potential is repulsive (attractive). (ii) When the de Broglie wavelength is much longer than the range of the potential, \( \delta_l \) is proportional to \( k^{2l+1} \). Find the proportionality constant.

3. **Scattering by an impenetrable sphere** (Sakurai 7.6): Consider the scattering of a particle by an impenetrable sphere: \( V(r) = 0 \) for \( r > a \) and \( V(r) = \infty \) for \( r < a \).
   a. Derive an expression for the s-wave \( (l = 0) \) phase shift. (You need not know the detailed properties of the spherical Besel functions to be able to do this simple problem.)
   b. What is the total cross section \( \sigma_{\text{tot}} \) in the extreme low-energy limit, \( k \to 0 \). Compare your answer with the geometric cross section \( \pi a^2 \).