Solutions to problems labeled “CL” may be found in the text.

1. CL 3.14
2. CL 3.17
3. CL 3.18
4. (a) CL 3.11
   (b) This problem is actually false. For instance, let \( L \) be the empty language (with equality). Then \( L \)-structures are simply sets, and are completely determined up to isomorphism by their cardinality. Hence all countably infinite models of the theory are isomorphic, but the theory is not complete, since a finite model with \( n \) elements will satisfy the formula which says there are exactly \( n \) elements, whereas a model of a different cardinality (e.g. an infinite model) will not.

What I intended to say was that the theory was consistent but had no finite models (so that it necessarily has an infinite model). In this case, if the theory were not complete there is some sentence \( F \) such that neither \( F \) nor \( \neg F \) is a consequence of \( T \), and \( T \) must then have two countable models \( M \) and \( N \) with \( M \models F \) and \( N \models \neg F \), so that \( M \) and \( N \) can not be isomorphic.

5. CL 3.21
6. (a) Consider the two sentences:

\[
F : \exists x \forall y (x < y) \\
G : \exists x \forall y (y < x)
\]

which say, respectively, that there is a least or greatest element. Neither of these sentences or their negations is a consequence of \( T \), so that the theory is not complete. We can in fact have any of the four possibilities of \( F \) and \( G \) being true or false. Now, the same method as used in part (c) below will show that once we decide whether there are least or greatest elements, that any two infinite models must be isomorphic. The fourth axiom implies that if there are at least two elements in a model, then in fact the model must be infinite. The only way that a model may fail to have two elements is if in fact there is a least element and a greatest element and they are equal. So there are five possible completions, with the following countable models (which are unique up to isomorphism):

<table>
<thead>
<tr>
<th>Theory</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T \cup {\neg F, \neg G} )</td>
<td>( \langle \mathbb{Q}, &lt; \rangle )</td>
</tr>
<tr>
<td>( T \cup {\neg F, G} )</td>
<td>( \langle \mathbb{Q} \cap (-\infty, 0), &lt; \rangle )</td>
</tr>
<tr>
<td>( T \cup {F, \neg G} )</td>
<td>( \langle \mathbb{Q} \cap [0, \infty), &lt; \rangle )</td>
</tr>
<tr>
<td>( T \cup {F, G, \exists x \exists y (x \neq y)} )</td>
<td>( \langle \mathbb{Q} \cap [0, 1), &lt; \rangle )</td>
</tr>
<tr>
<td>( T \cup {F, G, \forall x \forall y (x = y)} )</td>
<td>( \langle {0}, &lt; \rangle )</td>
</tr>
</tbody>
</table>

The last is obviously complete, and the first four are complete by (the corrected version of) 4(b).

(b) This is straightforward to check; we mainly want to note that there is a model of \( T' \).
Let $\mathcal{M}$ and $\mathcal{N}$ have underlying sets $M = \{m_0, m_1, \ldots\}$ and $N = \{n_0, n_1, \ldots\}$. We will construct a series of finite partial functions $f_i$ for $i \geq 0$ from $M$ to $N$ such that each $f_i$ preserves the ordering, each $f_{i+1}$ is an extension of $f_i$ (i.e. the domain of $f_{i+1}$ contains the domain of $f_i$ and the two functions agree on the domain of $f_i$), and $m_i$ is in the domain of $f_i$ and $n_i$ is in the range of $f_i$. Once we have done this, we then set $f = \bigcup_{i \geq 0} f_i$, i.e. the function whose domain is the union of the domains of the $f_i$; since the functions extend one another this is well-defined. Then $f$ will be a bijection between $M$ and $N$ by our domain and range conditions, and since each individual function preserves the orderings, so will $f$. Hence $f$ will be an isomorphism.

So we construct the $f_i$. We set $f_0(m_0) = n_0$. Now we assume $f_i$ has been defined to meet the conditions and define $f_{i+1}$. We first add $m_{i+1}$ to the domain. If $m_{i+1}$ was already in the domain of $f_i$ we need do nothing, so assume it was not. Since the domain of $f_i$ is finite, it can be enumerated in increasing order as \( \{a_0 < M a_1 < M \cdots < M a_k\} \), and the range can be enumerated as \( \{b_0 < N b_1 < N \cdots < N b_k\} \). Since $f_i$ was assumed to preserve order we must have $f_i(a_j) = b_j$ for $0 \leq j \leq k$. Now consider how $m_{i+1}$ is related to the $a_j$. Either it is less than all of them, or greater than all of them, or there is some $j$ with $0 \leq j < k$ such that $a_j < M m_{i+1} < M a_{j+1}$. We can then pick some $n_p \in N$ which bears the same relationship, i.e. either $n_p$ is less than all the $b_j$’s, greater than them all, or $b_j < N n_p < N b_{j+1}$. The first two cases follow from the lack of endpoints, and the third from density. We then set $f_{i+1}(m_{i+1}) = n_p$. This continues to preserve the ordering. Reversing the roles of $M$ and $N$, we next find some $m_q$ bearing the same order relationship to what has been defined in the domain as $n_{i+1}$ does to what has been defined in the range, and set $f_{i+1}(m_q) = n_{i+1}$. We have added at most two points to the domain, so that it is still finite. This finishes the construction.

7. CL 2.4

8. CL 2.18