Math 116a Solutions to Homework # 4
Due: 11-01-01

Solutions to problems labeled “CL” may be found in the text.

1. CL 3.5

2. The idea here is that because there are only finitely many elements of \( M \), and only finitely many constants, functions, and relations, there will be only finitely many things we need to say about the structure to completely describe it, and these can all be combined into one sentence. Let \( M \) have cardinality \( n \) and let \( a_1, \ldots, a_n \) enumerate these elements. Let \( k \) be such that \( \bar{c}M = a_k \). Let \( \pi \) be a function from \( \{1, \ldots, n\} \) to \( \{1, \ldots, n\} \) such that \( \bar{f}M(a_i) = a_{\pi(i)} \) for each \( 1 \leq i \leq n \). Let \( I \subseteq \{1, \ldots, n\}^2 \) be such that \( (a_i, a_j) \in \bar{R}M \) iff \( (i, j) \in I \). Now let \( \sigma \) be the following sentence:

\[
\exists x_1 \ldots \exists x_n \left( \bigwedge_{i \neq j} x_i \neq x_j \land \left( \forall y \bigvee_{i=1, \ldots, n} y = x_i \right) \land c = x_k \land \bigwedge_{i=1, \ldots, n} f(x_i) = x_{\pi(i)} \land \bigwedge_{(i, j) \in I} R(x_i, x_j) \land \bigwedge_{(i, j) \notin I} \neg R(x_i, x_j) \right)
\]

This sentence asserts that there are precisely \( n \) elements, and they behave precisely as \( a_1, \ldots, a_n \). If we have any other model \( N \) with \( N \models \sigma \), then if we take a set of elements \( b_1, \ldots, b_n \in N \) which work for \( x_1, \ldots, x_n \), then it is easy to see that the map sending \( a_i \) to \( b_i \) for each \( i \) is an isomorphism of \( M \) with \( N \). Conversely, if \( N \) is isomorphic to \( M \) then \( N \models \sigma \), since (as was shown in class) the set of formulas modeled by an \( L \)-structure is preserved under isomorphism.

3. CL 3.12

4. (a) We note that since we have constants for \( 0 \) and \( 1 \), these must be contained in any substructure, and since we have symbols for \( \sim \), \( \prec \), and \( \neg \), any substructure must be closed under the Boolean algebra operations. Now, looking at the axioms for Boolean algebras, we see that they all assert that for every element or pair or triple of elements in the structure, some algebraic relationship is true. So if we have any elements in a substructure, they are also elements in the original structure. Since the relationship holds between them in the original structure, and all the functions in a substructure are restrictions of the original functions, this relation must also hold in the substructure.

(b) Similarly, we see that any substructure must contain the group identity and be closed under group multiplication and inverse. Also as before, all the axioms of a group assert that some relationship holds between elements (since we have a symbol for inverse, we do not have to assert that for each \( x \) there is an element \( y \) with \( x \cdot y = e \); we simply have to say that \( x \cdot x^{-1} = e \), so similar reasoning applies as in (a).

(c) Let \( G = (\mathbb{Z}, +) \) be the additive group of integers. Let \( H = (\mathbb{N}, +) \) be the additive semigroup of natural numbers. This is a substructure of \( G \), since the sum of two non-negative integers is still non-negative, and the structure needs only be closed under the group multiplication \( (+) \) in this language; however, it is not a group because it does not have inverses.

(d) The chief difference is that in (c), a substructure need not be closed under inverses, and it need not contain the identity element. In (b), all of the group axioms can be expressed as universal formulas, which we will see are preserved in substructures, whereas in (c) they can not be expressed as universal formulas.
5. (a) We find an automorphism $\varphi$ of the structure $\langle \mathbb{Z}, 0, 1, \cdot, \rangle$ such that the set of even integers is not invariant under $\varphi$. We set $\varphi(0) = 0$. For any positive $n$, let $n$ have prime factorization $2^{n_2} \cdot 3^{n_3} \cdot 5^{n_5} \cdots p^{n_p}$. We let $\varphi(n) = 2^{n_2} \cdot 3^{n_3} \cdot 5^{n_5} \cdots p^{n_p}$, i.e. we interchange factors of 2 and 3. We do the same for negative numbers. This is a bijection, it fixes 0 and 1, and it preserves multiplication, so that it is an automorphism of the structure, but it sends the set of even integers to the set of multiples of 3, so the set of even integers is not definable in the structure.

(b) We find an automorphism $\varphi$ of $\langle \mathbb{R}, 0, 1, \cdot, < \rangle$ so that the set of integers is not invariant under $\varphi$. Let $\varphi(x) = x^3$. Then $\varphi$ is a bijection of $\mathbb{R}$ with itself, $\varphi(0) = 0$, $\varphi(1) = 1$, $(x \cdot y)^3 = x^3 \cdot y^3$, and $x < y$ iff $x^3 < y^3$, so $\varphi$ is an automorphism of the structure. The set of integers is not preserved under $\varphi$: Although the cube of integer is an integer, not every integer is the cube of an integer, so that although we do have $x \in \mathbb{Z} \Rightarrow \varphi(x) \in \mathbb{Z}$, we do not have $\varphi(x) \in \mathbb{Z} \Rightarrow x \in \mathbb{Z}$. Note that the automorphism $\varphi$ here is definable in the structure: $\varphi(x) = y$ iff $\langle \mathbb{R}, 0, 1, \cdot, < \rangle \models y = x \cdot x \cdot x$.

6. (a) First, since we have $a \sim (a \sim b) = a$ and $b \sim (a \sim b) = b$, we have that $a \leq a \sim b$ and $b \leq a \sim b$ so that $a \sim b$ is an upper bound of $a$ and $b$. Suppose $c$ is another upper bound of $a$ and $b$. Then $c \sim a = a$ and $c \sim b = b$, so that we have $c \sim (a \sim b) = (c \sim a) \sim (c \sim b) = a \sim b$, so that $a \sim b \leq c$; so $a \sim b$ is a least upper bound. Similarly, we have $(a \sim b) \sim a = a \sim b$ so that $a \sim b \leq a$; so $a \sim b \leq b$ is identical. If we have $c$ with $c \leq a$ and $c \leq b$, then $c \sim (a \sim b) = c \sim a \sim b = (c \sim a) \sim (c \sim b) = c \sim c = c$, so that $c \leq a \sim b$, so $a \sim b$ is a greatest lower bound of $a$ and $b$.

(b) Let $b \neq 0$. If $b$ is an atom, we are done. Otherwise there is some element $b'$ with $0 < b' < b$. Now if $b'$ is an atom we are done; otherwise there is a $b''$ with $0 < b'' < b' < b$. We continue in this fashion; since the Boolean algebra is finite and all of these elements are distinct this cannot go on forever and we must end with some $b^{(n)} \leq b$ which is an atom.

(c) Consider $a \sim -b$. Since $(a \sim -b) \sim b = a \sim (b \sim -b) = 0$, this is only $\leq b$ if it equals $0$; so if $a \sim -b \neq 0$ then any atom $c \leq a \sim -b$ will be below $a$ but not below $b$. Similarly, if $b \sim -a \neq 0$, then any atom $c \leq b \sim -a$ will be below $b$ but not below $a$. So it will suffice to show that at least one of $a \sim -b$ and $b \sim -a$ is not $0$.

We will show that $a \sim -b = 0$ iff $a \leq b$ (so also $b \sim -a = 0$ iff $b \leq a$), so that if both of these elements are $0$ we must have $a \leq b$ and $b \leq a$, i.e. $a = b$. We have $a \leq b$ iff $a \sim b = b$ by definition. If $a \sim b = a$ then $a \sim b \sim -b = a \sim -b$ so $0 = a \sim -b$. Conversely, if $a \sim -b = 0$, then $b = b \sim 0 = b \sim (a \sim -b) = (b \sim a) \sim (b \sim -b) = (b \sim a) \sim 1 = b \sim a$, and we saw last time that $a \sim b = b$ was equivalent to $a \leq b$, so we are done.

(d) Define $f$ as in the hint. Then part (c) shows that $f$ is injective. To see that $f$ is surjective, let $B \subseteq X$ be non-empty (clearly $f(0) = 0$), $B = \{b_1, \ldots, b_n\}$. Let $a = b_1 \sim b_1 \sim \cdots \sim b_n \in \mathcal{A}$. We claim that $f(a) = B$. For this it suffices to check that if $b \leq a$ is an atom, then $b \in B$. If $b \leq a$ then $b \sim (b_1 \cdots \sim b_n) = b$, so $(b \sim b_1) \cdots \sim (b \sim b_n) = b$. Each of the terms $b \sim b_i$ is $\leq b$, and since $b$ is an atom it is either $b$ or $0$. If all the terms are $0$, the whole meet is $0$, so there is some $b_i$ with $b \sim b_i = b$. But this says $b \leq b_i$; and $b_i$ is an atom this means $b = b_i$ (since $b \neq 0$). So $b \in B$ as we wished.

We next check that $f(-a) = X \setminus f(a)$. If $b$ is an atom with $b \in f(-a)$, then $b \leq -a$, so $b \not\leq a$ (or else $b \leq a \sim -a = 0$, which is impossible). Thus $b \not\in f(a)$. Conversely, if $b \not\in f(a)$, then $b \leq a$. But then we have $b \leq a$, since $a = b \sim 1 = b \sim (a \sim -a) = (b \sim a) \sim (b \sim -a)$; each of these is $\leq b$ so at least one must equal $b$. Thus $b \in f(-a)$.

Next we check that $f(a \sim b) = f(a) \cap f(b)$. If $c \in f(a) \cap f(b)$, then $c \in f(a)$ so $c \leq a$, and $c \in f(b)$ so $c \leq b$; thus $c \leq a \sim b$ by part (a). If $c \in f(a \sim b)$, then $c \leq a \sim b$, so we saw last time that $c \sim b = b$ was equivalent to $c \leq b$, so we are done.

Preservation of $\sim$ is essentially the same.