Problems labeled “CL” are from the text.

1. CL 3.14
2. CL 3.17
3. CL 3.18
4. (a) CL 3.11
   Note that if all of the models of a theory are isomorphic, they must all have the same cardinality. We will see later that this implies that the theory has only finite models.
   (b) Show that if $T$ is a theory with at least one countably infinite model, and all of the countably infinite models of $T$ (those whose underlying set is bijective with the set of natural numbers) are isomorphic, then $T$ is complete.
5. CL 3.21
6. Let $\mathcal{L} = \langle < \rangle$ have a single binary relation symbol, and consider the theory $T$ of dense linear orders whose axioms are the following:
   \[
   \neg \exists x \exists y (x < y \land y < x) \\
   \forall x \forall y \forall z ((x < y \land y < z) \Rightarrow x < z) \\
   \forall x \forall y (x = y \lor x < y \lor y < x) \\
   \forall x \forall y (x < y \Rightarrow \exists z (x < z \land z < y))
   \]
   These are the linear orders where no element has an immediate successor or predecessor.
   (a) Show that $T$ is not complete. [Hint: Consider whether there can be least or greatest elements.]
   How many inequivalent complete extensions of $T$ are there?
   (b) Let $T'$ be the theory of dense linear orders without endpoints, i.e. $T$ together with with the sentence:
   \[
   \forall x \exists y \exists z (x < y \land z < x)
   \]
   Note that this theory has no finite models. Show that $\langle \mathbb{Q}, < \rangle$ is a model of $T'$, so that $T'$ is consistent.
   (c) Show that the theory of dense linear orders without endpoints is countably categorical, i.e. if $\mathcal{M}$ and $\mathcal{N}$ are two countable models of $T'$, then $\mathcal{M} \cong \mathcal{N}$ (and so both are isomorphic to $\langle \mathbb{Q}, < \rangle$). [Hint: Construct an isomorphism using what is called a “back-and-forth” argument. Enumerate the elements of the underlying sets of $\mathcal{M}$ and $\mathcal{N}$. Define progressively larger partial isomorphisms in such a way that elements are alternately added to the domain and range of the maps to achieve a bijection, and use density and lack of endpoints to ensure that this can be done indefinitely.]
7. CL 2.4
8. CL 2.18