1. Let $L = \langle 0; S, +, \times; < \rangle$ be the Language of Arithmetic, where $0$ is a constant, $S$ is a unary function, $+$ and $\times$ are binary functions, and $<$ is a binary relation. The standard model of arithmetic is $\mathcal{N} = \langle \mathbb{N}; 0; S, +, \times; < \rangle$, where the function $S$ expresses the successor of a natural number, and the other symbols have the natural intended meaning. Translate each of the following English statements into a formula in the language of arithmetic:

(a) $x_1$ is divisible by $x_2$.
(b) $x$ is odd.
(c) No natural number is both even and odd.
(d) Every natural number is the sum of four squares (of natural numbers).
(e) The number 5 is prime.
(f) There is no natural number whose square is negative.

2. Let $L = \langle e, \cdot, -1 \rangle$ be the language of groups, where $e$ is a constant (the identity), $\cdot$ is a binary function (the group multiplication) and $-1$ is a unary function (inverse). Translate each of the following formulas into a simple English expression:

(a) $\forall x \forall y x \cdot y = y \cdot x$
(b) $\forall x \exists y y^{-1} = x$
(c) $\neg \exists y y \cdot y = x$
(d) $\forall x x \cdot x = x \Rightarrow x = e$

3. For each of the following formulas, determine which occurrences of variables are free and bound, and what the free variables of each formula are. Determine which of the formulas are closed. $P$, $Q$, and $R$ are relation symbols; $a$ and $b$ are constant symbols.

(a) $\forall x_1 \exists x_2 P(x_1, x_2)$
(b) $\forall x_2 (R(x_1) \Rightarrow \exists x_1 Q(x_1))$
(c) $R(x_1, a) \lor R(x_2, b)$
(d) $\forall x_1 R(x_1, x_2) \lor \exists x_2 (R(x_1, x_2) \land Q(x_2))$
(e) $\forall x_1 \exists x_2 P(x_1, x_2)$
(f) $\forall x_1 (P(x_1, x_2) \Rightarrow \forall x_2 Q(x_1))$

4. CL 3.4

5. CL 3.7
The next problems are an introduction to Boolean Algebras (see chapter 2 of the text). The definition we
give here is somewhat different from the definition there (but all of the concepts in one presentation can be
defined in terms of those of the other).

A Boolean Algebra is the structure of the form $A = (A, ∨, , ¬, 0, 1)$ where $A$ is a set, , and $∨$ are binary functions on $A$, $¬$ is a unary function on $A$, and $0$ and $1$ are distinguished elements, which satisfies the following axioms:

1. $0 ≠ 1$
2. $∀x (0 ∨ x = 0 ∧ 0 ∨ x = x ∧ 1 ∨ x = x ∧ 1 ∨ x = 1)$
3. $∀x∀y (x ∨ y = y ∨ x ∧ x ∨ y = y ∨ x)$
4. $∀x∀y∀z ((x ∨ y) ∨ z = x ∨ (y ∨ z) ∧ (x ∨ y) ∨ z = x ∨ (y ∨ z))$
5. $∀x∀y∀z (x ∨ (y ∨ z) = (x ∨ y) ∨ z ∧ x ∨ (y ∨ z) = (x ∨ y) ∨ z)$
6. $∀x∀y (x ∨ (x ∨ y) = x ∧ x ∨ (x ∨ y) = x)$
7. $∀x (x ∨ ¬x = 0 ∧ x ∨ ¬x = 1)$

(These axioms should seem familiar)

The next three problems give some examples:

6. Let $A = \{0, 1\}$, where $0$, $1$, and $¬$ are given by the usual propositional connectives $∨$, $∧$, and $¬$. Show that this is a Boolean algebra (this is the smallest Boolean algebra, sometimes denoted $A_0$).

7. Let $X$ be any non-empty set and let $A = ℙ(X)$ be the power set of $X$: $ℙ(X) = \{Y : Y \subseteq X\}$. Let $0$, $1$, and $¬$ be given by union, intersection, and complementation (relative to $X$), and let $0 = \emptyset$ and $1 = X$. Show that this is a Boolean algebra (such an algebra is called an algebra of sets).

8. Recall that for a formula $F ∈ ℱ$, $[F]$ denotes the equivalence class of $F$ under the relation $∼$ of logical equivalence. We denote the set of equivalence classes by $ℱ/∼$. Let $A$ be the set $ℱ/∼$. Let $0$ be the equivalence class consisting of antilogies and $1$ the class consisting of tautologies, and set $[F] ∼ [G] = [F ∨ G], [F] ∨ [G] = [F ∧ G]$, and $¬[F] = [¬F]$.

(a) Show that these operations are well-defined, i.e. if $F_1 ∼ F_2$ and $G_1 ∼ G_2$, we have $[F_1 ∨ G_1] = [F_2 ∨ G_2], [F_1 ∧ G_1] = [F_2 ∧ G_2]$, and $[¬F_1] = [¬F_2]$.

(b) Show that with these operations $ℱ/∼$ is a Boolean algebra. This is called the Lindenbaum Algebra of propositional logic.

9. Given a Boolean algebra we can define a relation $≤$ on $A$ by setting

$$a ≤ b ⇔ a ∨ b = a$$

(a) Show that this is an order relation. Is it always a linear order?

(b) Show that $a ≤ b ⇔ a ∨ b = b$.

(c) In each of the three examples above, determine what this ordering is (i.e. give a natural definition of the order that does not involve the Boolean algebra operations).