IN ALL PROBLEMS BELOW YOU CANNOT USE THE COMPLETENESS THEOREM
(since the exercises below form parts of the proof of that theorem). ALSO HERE “FOR-
MULA” OR “WFF” MEANS “WELL-FORMED FORMULA IN PROPOSITIONAL LOGIC
BUILT USING PROPOSITIONAL VARIABLES, PARENTHESES, AND ONLY ¬, ⇒.”

(40%) 1. (i) Show that for any formula \( A \),

\[ \vdash (\neg\neg A \Rightarrow A) . \]

(ii) Prove the following in propositional logic:

Let \( S \) be any set of wffs and \( A \) any wff. If \( S \cup \{ A \} \) is formally inconsistent, then
\[ S \vdash \neg A . \]

(iii) Show that if \( S \) is formally inconsistent, then \( S \vdash A \) for any wff \( A \).

(20%) 2. Prove the following in propositional logic::

Let \( S \) be any set of wffs and \( A, B \) any wffs. Then:

\[ S \cup \{ A \} \vdash \neg B \iff S \cup \{ B \} \vdash \neg A . \]

(40%) 3. (i) Show that

\[ \neg(A \Rightarrow B) \vdash A \]

and

\[ \neg(A \Rightarrow B) \vdash \neg B \]

(ii) Prove the following in propositional logic::

If \( \bar{S} \) is a formally consistent and complete set of formulas and \( \nu \) is the valuation defined by

\[ \nu(p_i) = \begin{cases} 
1, & \text{if } p_i \in \bar{S} \\
0, & \text{if } p_i \notin \bar{S},
\end{cases} \]

then for any formula \( A \),

\[ \nu(A) = 1 \iff A \in \bar{S} . \]