(30%) 1. (i) Let $A$ be a wff and assume that the only connectives appearing in $A$ are among $\neg, \land, \lor$ (i.e., $\Rightarrow, \Leftrightarrow$ don’t appear). Let $A^*$ be obtained from $A$ by replacing each propositional variable $p$ appearing in $A$ by $\neg p$ and replacing $\land$ by $\lor$ and $\lor$ by $\land$. Show that

$$\neg A \equiv A^*$$

(i.e., $\neg A, A^*$ are logically equivalent).

(ii) Suppose $A$ is a wff as in (i). Let $A'$ be the wff obtained from $A$ by replacing $\land$ by $\lor$ and $\lor$ by $\land$. We call $A'$ the dual of $A$. (Example: $(p \lor q) \land \neg r$ is the dual of $(p \land q) \lor \neg r$.) Show that $A$ is a tautology iff $\neg A'$ is a tautology.

(iii) (Principle of duality) For $A, B$ wff as in (ii), show that

$$A \equiv B \iff A' \equiv B'.$$

(20%) 2. Consider the wff

$$A_n = ((\ldots (p_1 \Leftrightarrow p_2) \Leftrightarrow p_3) \Leftrightarrow \ldots) \Leftrightarrow p_n).$$

Show that a valuation $v$ satisfies $A_n$ exactly when $v(p_i) = 0$ for an even number of $i$ in the interval $1 \leq i \leq n$.

(20%) 3. For each $n = 2, 3, 4, \ldots$ find a set $S = \{A_1, A_2, \ldots, A_n\}$ consisting of $n$ wff such that $S$ is not satisfiable, but any proper nonempty subset $S' \subseteq S$ is satisfiable.

(30%) 4. A set $S$ of wff is independent if for any wff $A \in S$, $S \setminus \{A\} \not\models A$, i.e., $A$ is not implied logically by the rest of the wff in $S$. (So, by definition, the empty set $\emptyset$ is independent, and $S = \{A\}$ is independent iff $A$ is not a tautology.)

(i) Which of the sets

(a) $\{p \Rightarrow q, q \Rightarrow r, r \Rightarrow q\}$

(b) $\{p \Rightarrow q, q \Rightarrow r, p \Rightarrow r\}$

(c) $\{p \Rightarrow r, r \Rightarrow q, q \Rightarrow p, r \Rightarrow (q \Rightarrow p)\}$

are independent and which are not?

(ii) Two sets of wff, $S, S'$ are called equivalent if $S \models A'$ for any $A' \in S'$ and $S' \models A$ for any $A \in S$. (So, by definition, if $S = \{A\}$, where $A$ is a tautology, $\emptyset$ is equivalent to $S$.) Show that for any finite set $S$ of wff, there is a subset $S' \subseteq S$ which is independent and equivalent to $S$. 

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