Consider the following two surfaces:

1. \( \mathbb{H}^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\} \) with the metric \( \frac{1}{x_2^2}(dx_1^2 + dx_2^2) \) or, in the notation of the first fundamental form, \( E = G = \frac{1}{x_2^2} \) and \( F = 0 \)

2. \( \mathbb{D}^2 = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\} \) with the metric \( \left(\frac{2}{1-x_1^2-x_2^2}\right)^2 (dx_1^2 + dx_2^2) \) or, in the notation of the first fundamental form, \( E = G = \left(\frac{2}{1-x_1^2-x_2^2}\right)^2 \) and \( F = 0 \).

**Theorem 1.** Let \( e_2 = (0,1) \). The map

\[
i(x) = 2 \frac{x + e_2}{\|x + e_2\|} - e_2
\]

is an isometry of \( \mathbb{D}^2 \) with \( \mathbb{H}^2 \).

**Proof.** A lengthy computation. \( \square \)

\( \mathbb{D}^2 \) and \( \mathbb{H}^2 \) define the same abstract surface, which is called the hyperbolic plane. \( \mathbb{H}^2 \) is called the upper half plane model and \( \mathbb{D}^2 \) is called the Poincaré disc model. From the formula in Do Carmo, exercise 4-3.2 it is easy to see that the hyperbolic plane has constant Gauss curvature \( K = -1 \).

The next step will be to find all geodesics of \( \mathbb{H}^2 \). We use the following lemma:

**Lemma 2.** If \( \varphi : S \to S \) is an isometry, and \( C = \text{Fix}(\varphi) \) is a curve, then \( C \) is a geodesic.
Proof. Let \( p \in C \) and \( v \in T_pS \) be a vector tangent to \( C \), and call \( \alpha_{p,v} \) the unique geodesic starting at \( p \) with velocity \( v \). Then we know from the properties of geodesics that \( \varphi \circ \alpha_{p,v} = \alpha_{\varphi(p),d_p\varphi(v)} \). Since \( C \) consists of fixed points for \( \varphi \) we have \( \varphi(p) = p \) and \( d_p\varphi(v) = v \) so \( \varphi \circ \alpha_{p,v} = \alpha_{p,v} \). This implies that \( \alpha_{p,v} \) is a parameterisation of \( C \).

Theorem 3. The geodesics of \( \mathbb{H}^2 \) are straight vertical half-lines and half circles with centre on the \( x_1 \)-axis.

Proof. The reflection around the \( x_1 \)-axis \( \sigma(z) = -\bar{z} \) and the inversion \( \tau(z) = \frac{1}{\bar{z}} \) (here we identify \( \mathbb{H}^2 = \{ z \in \mathbb{C} : \Im z > 0 \} \) are isometries of \( \mathbb{H}^2 \), therefore \( \text{Fix}(\sigma) = \{(0,t) : t > 0\} \) and \( \text{Fix}(\tau) = \{ z \in \mathbb{H}^2 : |z| = 1 \} \) are geodesics.

Dilations with centre 0 (\( z \mapsto az \) with \( a > 0 \)) and translations parallel to the \( x_1 \)-axis (\( z \mapsto z + b \) with \( b \in \mathbb{R} \)) are isometries of \( \mathbb{H}^2 \), and send circles to circles and straight lines to straight lines, therefore all vertical straight lines and all half circles with centre on the \( x_1 \)-axis are geodesic. Moreover for every point \( p \in \mathbb{H}^2 \) and every \( v \in T_p\mathbb{H}^2 \) there is either a vertical straight line or a half circle with centre on the \( x_1 \)-axis which passes through \( p \) and is tangent to \( v \). Since there is a unique geodesic for any given initial data, there are no more geodesics in \( \mathbb{H}^2 \) than those we have already found.

If we say that two geodesics are parallel if they do not intersect, we see that geodesics in \( \mathbb{H}^2 \) satisfy all properties of straight lines in the Euclidean plane, except those depending on the axiom of parallels (for historical reasons also called Euclid’s fifth axiom, even if you need more axioms than Euclid’s five to give solid foundations to Euclidean geometry). With the construction of \( \mathbb{H}^2 \) we have embedded a model of a non-Euclidean plane in the Euclidean plane, and this shows that the axiom of parallels is independent of the other ones.

Definition 4. Let \( l_1 \) and \( l_2 \) be two distinct geodesics in \( \mathbb{H}^2 \). Then three things can happen:

1. \( l_1 \cap l_2 \neq \emptyset \)
2. \( l_1 \cap l_2 = \emptyset \) but \( \sup\{d(p,q) : (p, q) \in l_1 \times l_2\} = 0 \)
3. \( d(p,q) > d_0 > 0 \) for all \( p \in l_1 \) and \( q \in l_2 \).
In the first case we say that $l_1$ and $l_2$ are incident, in the second one that they are asymptotically parallel, and in the third one that they are ultraparallel. In the half plane model asymptotically parallel geodesics correspond to geodesics converging to the same boundary point, and ultraparallel geodesics corresponds to geodesics with distinct limit points. Let’s prove this claim. If $l_1$ and $l_2$ converge to the same boundary point we can assume, up to isometry, that they converge to $\infty$, so they are vertical straight lines. Then it is easy to find sequences of points $p_i \in l_1$ and $q_i \in l_2$ with $d(p_i, q_i) \to 0$. Assume now the existence of the two sequences. Clearly they cannot converge in $\mathbb{H}^2$ otherwise the limit point would be an intersection point between $l_1$ and $l_2$. Then up to taking subsequences $p_i$ and $q_i$ converge to points $p_\infty$ and $q_\infty$ on the boundary of $\mathbb{H}$. Then the arcs of geodesic from $p_i$ to $q_i$ converge to the geodesic with limit points $p_\infty$ and $q_\infty$, which has infinite length.

Let us denote by $Isom^+(\mathbb{H}^2)$ the group of orientation preserving isometries of $\mathbb{H}^2$, and by $Conf^+(\mathbb{H}^2)$ the group of orientation preserving conformal transformations of $\mathbb{H}^2$. A linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which preserves the angles and is orientation preserving is a positive multiple of a rotation, and so it is complex linear. This proves that $Conf^+(\mathbb{H}^2)$ consists of all biholomorphism of $\mathbb{H}^2$. It is well known from complex analysis that

$$Conf^+(\mathbb{H}^2) = \left\{ z \mapsto \frac{az + b}{cz + d} \text{ with } a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}.$$  

**Theorem 5.** $Isom^+(\mathbb{H}^2) \cong Conf^+(\mathbb{H}^2)$

**Proof.** It’s not hard to see that dilations ($z \mapsto az$), translations ($z \mapsto z + b$), and $z \mapsto -\frac{1}{z}$ are isometries of $\mathbb{H}^2$ and generate $Conf^+(\mathbb{H}^2)$. This proves $Conf^+(\mathbb{H}^2) \subset Isom^+(\mathbb{H}^2)$. The Hyperbolic metric is conformally equivalent to the Euclidean metric, so transformations of $Conf^+(\mathbb{H}^2)$ are conformal for the hyperbolic metric too. This proves that $Isom^+(\mathbb{H}^2) \subset Conf^+(\mathbb{H}^2)$ because isometries are conformal maps, hence the equality.\hfill $\Box$

We apply our knowledge of hyperbolic isometries to understand hyperbolic circles. We can see a circle with centre $p$ (the set of points at the same distance from $p$) as an orbit of the stabiliser of $p$. Choose $p = i$, then it is easy to see that the orientation preserving isometries fixing $i$ can be represented by matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$. Those matrices can be
written as \(\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}\), and the orbit of the point \(ri\), with \(r > 0\), is
\[
ri \cos \alpha + \sin \alpha \\
-ri \sin \alpha + \cos \alpha.
\]
Now let’s compute
\[
\frac{rai + b}{-rbi + a} - \frac{r^2 + 1}{2r}i = \frac{(r^2 - 1)ai - (r^2 - 1)rb}{2r(a - rbi)} = \frac{i(r^2 - 1)(a + rbi)}{2r(a - rbi)}.
\]
So the hyperbolic circle with centre \(i\) passing through \(ri\) is the Euclidean circle with centre \(\frac{r^2 + 1}{2r}\) and radius \(\frac{r^2 - 1}{2r}\).

Now we can move the centre around using translations and dilations, which preserve circles, and we obtain the following theorem.

**Theorem 6.** Hyperbolic circles are Euclidean circles, but with different centre and radius.

**Definition 7.** A Hyperbolic metric on a surface \(S\) is a metric (i.e. a first fundamental form) which is locally isometric to the hyperbolic plane \(\mathbb{H}^2\).

**Theorem 8.** A compact, connected, oriented surface \(S\) without boundary with genus \(g\) (i.e. \(S\) is a doughnut with \(g\) holes) admits a hyperbolic metric if and only if \(g > 1\).

**Proof.** A hyperbolic metric has constant Gauss curvature \(K = -1\) because Gauss curvature is invariant under local isometries, so if \(S\) admits a hyperbolic metric, then Gauss Bonnet Theorem implies that
\[
\chi(S) = 2 - 2g < 0.
\]
From this we obtain \(g > 1\). A surface \(S\) with genus \(g \geq 1\) can be obtained by identifying the edges of a \(4g\)-gone two by two. The identification is made so that all vertices of the \(4g\)-gone are identified to the same point in \(S\). If we can find a hyperbolic regular \(4g\)-gone with the sum of interior angle \(2\pi\) (this last condition is needed because all vertices go to the same point), then we can identify the opposite sides via hyperbolic isometries, and therefore we can put a hyperbolic metric on \(S\).

The procedure to construct the \(4g\)-gone is the following. (It is best seen in the disc model \(\mathbb{D}^2\), but it works in the upper half plane model too.) Take a very small regular \(4g\)-gone with geodesic boundary. It will almost look like a Euclidean \(4g\)-gone, and therefore the sum of its interior angles is \(> 2\pi\).
provided that $g > 1$. Then start enlarging the $4g$-gone continuously while keeping it regular. In the limit, when the vertices of the $4g$-gone reach the boundary, the sum of the interior angles will become 0. Since the sum of the interior angles depends continuously on the $4g$-gone, we find a $4g$-gone whose sum of interior angles is $2\pi$. □