The Method of Frobenius

Consider the equation

\[ x^2 y'' + xp(x)y' + q(x)y = 0, \]

(1)

where \( x = 0 \) is a regular singular point. Then \( p(x) \) and \( q(x) \) are analytic at the origin and have convergent power series expansions

\[ p(x) = \sum_{k=0}^{\infty} p_k x^k, \quad q(x) = \sum_{k=0}^{\infty} q_k x^k, \quad |x| < \rho \]

(2)

for some \( \rho > 0 \). Let \( r_1, r_2 \) \((\mathbb{R}(r_1) \geq \mathbb{R}(r_2))\) be the roots of the indicial equation

\[ F(r) = r(r - 1) + p_0 r + q_0 = 0. \]

(3)

Depending on the nature of the roots, there are three forms for the two linearly independent solutions on the intervals \( 0 < |x| < \rho \). The power series that appear in these solutions are convergent at least in the interval \( |x| < \rho \). (Proof: Coddington)

**Case 1: Distinct roots not differing by an integer \((r_1 - r_2 \neq N, N \in \mathbb{Z})\)**

The two solutions have the form

\[ y_1(x) = x^{r_1} \sum_{k=0}^{\infty} a_k(r_1) x^k \]

(4)

\[ y_2(x) = x^{r_2} \sum_{k=0}^{\infty} b_k(r_2) x^k \]

(5)

where \( a_k(r_1) \) and \( b_k(r_2) \) are determined by substitution of (4) or (5) into equation (1) to obtain the corresponding recurrence relation.

**Case 2: Repeated root \((r_1 = r_2)\)**

The first solution \( y_1(x) \) has form (4) and the second solution has the form

\[ y_2(x) = y_1(x) \log x + x^{r_1} \sum_{k=1}^{\infty} b_k(r_1) x^k. \]

(6)

Note that the term \( k = 0 \) is omitted as it would just give a multiple of \( y_1(x) \).

**Case 3: Roots differing by an integer \((r_1 - r_2 = N, N \in \mathbb{Z}^+)\)**

The first solution \( y_1(x) \) has form (4) and the second solution has the form

\[ y_2(x) = cy_1(x) \log x + x^{r_2} \sum_{k=0}^{\infty} b_k(r_2) x^k. \]

(7)

where \( c \) may turn out to be zero. The constant \( b_N(r_2) \) is arbitrary and may be set to zero. This is evident by writing

\[ x^{r_2} \sum_{k=0}^{\infty} b_k(r_2) x^k = b_0 x^{r_2} + \ldots + b_{N-1} x^{r_2+N-1} + x^{r_1} \left( b_N + b_{N+1} x + b_{N+2} x^2 + \ldots \right) \]

form of \( y_1(x) \)

(8)

so we see that \( b_N(r_2) \) plays the same role as \( a_0(r_1) \) and merely adds multiples of \( y_1(x) \) to \( y_2(x) \).
Example: Case 1

Consider

\[ 4xy'' + 2y' + y = 0 \]  \hspace{1cm} (9)

so \( x = 0 \) is a regular singular point with \( p(x) = \frac{1}{2} \) and \( q(x) = \frac{1}{4} \). The power series in \( y_1 \) and \( y_2 \) will converge for \( |x| < \infty \) since \( p \) and \( q \) have convergent power series in this interval. By (3), the indicial equation is

\[ r(r - 1) + \frac{1}{2}r = 0 \implies r^2 - \frac{1}{2}r = 0 \]  \hspace{1cm} (10)

so \( r_1 = \frac{1}{2} \) and \( r_2 = 0 \) (Note: \( p_0 = \frac{1}{2}, q_0 = 0 \)). Substituting \( y = x^r \sum_{k=0}^{\infty} a_k x^k \) into (9) and shifting the indices of the first two series so all terms are of form \( x^{k+r} \) we get

\[ 4 \sum_{k=-1}^{\infty} (k + r + 1)(k + r)a_{k+1}x^{k+r} + 2 \sum_{k=-1}^{\infty} (k + r + 1)a_{k+1}x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r} = 0. \]  \hspace{1cm} (11)

All coefficients of powers \( x^{k+r} \) must equate to zero to obtain a solution. The lowest power is \( x^{r_1+1} \) for \( k = -1 \) and this yields

\[ 4r(r - 1) + 2r = 0 \implies r^2 - \frac{1}{2}r = 0 \]  \hspace{1cm} (12)

which is just the indicial equation as expected. For \( k \geq 0 \), we obtain

\[ 4(k + r + 1)(k + r)a_{k+1} + 2(k + r + 1)a_{k+1} + a_k = 0 \]  \hspace{1cm} (13)

corresponding to the recurrence relation

\[ a_{k+1} = \frac{-a_k}{(2k + 2)(2k + 2 + 1)}, \quad k = 0, 1, 2... \]  \hspace{1cm} (14)

**First Solution:** To find \( y_1 \) apply (14) with \( r = r_1 = \frac{1}{2} \) to get the recurrence relation

\[ a_{k+1} = \frac{-a_k}{(2k + 3)(2k + 2)}, \quad k = 0, 1, 2... \]  \hspace{1cm} (15)

Then

\[ a_1 = \frac{-a_0}{3 \cdot 2}, \quad a_2 = \frac{-a_1}{5 \cdot 4}, \quad a_3 = \frac{-a_2}{7 \cdot 6}, \quad ... \]  \hspace{1cm} (16)

so

\[ a_1 = \frac{-a_0}{3!}, \quad a_2 = \frac{a_0}{5!}, \quad a_3 = \frac{-a_0}{7!}, \quad ... \]  \hspace{1cm} (17)

Since \( a_0 \) is arbitrary, let \( a_0 = 1 \) so

\[ a_k(r_1) = \frac{(-1)^k}{(2k + 1)!}, \quad k = 0, 1, 2... \]  \hspace{1cm} (18)

and

\[ y_1(x) = x^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^k. \]  \hspace{1cm} (19)

**Second Solution:** To find \( y_2 \), just apply (14) with \( r = r_2 = 0 \) to get the recurrence relation

\[ b_{k+1} = \frac{-b_k}{(2k + 2)(2k + 1)}. \]  \hspace{1cm} (20)

Letting the arbitrary constant \( b_0 = 1 \), then

\[ b_k(r_2) = \frac{(-1)^k}{(2k)!} \]  \hspace{1cm} (21)
so
\[ y_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^k. \]  

Example: Case 2

Consider
\[ Ly = xy'' + y' - y = 0 \] (23)
with \( p(x) = 1 \) and \( q(x) = -x \) and a regular singular point at \( x = 0 \). The power series in \( y_1 \) and \( y_2 \) will converge for \( |x| < \infty \) since \( p \) and \( q \) have convergent power series in this interval. The indicial equation is given by
\[ r(r - 1) + r = 0 \Rightarrow r^2 = 0 \] (24)
so \( r_1 = r_2 = 0 \).

First solution: Substituting \( y = \sum_{k=0}^{\infty} a_k x^k \) into (23) results in
\[ \sum_{k=0}^{\infty} (k + 1)k a_{k+1} x^k + \sum_{k=0}^{\infty} (k + 1) a_{k+1} x^k - \sum_{k=0}^{\infty} a_k x^k = 0 \] (25)
after shifting indices in the first two series to express all terms as multiples of \( x_k \). Regrouping terms gives
\[ \sum_{k=0}^{\infty} [(k + 1)k a_{k+1} + (k + 1) a_{k+1} - a_k] x^k = 0, \] (26)
so equating all coefficients of powers of \( x \) to zero gives
\[ a_{k+1} = \frac{a_k}{(k + 1)^2}, \quad k \geq 0. \] (27)
Then for \( k \geq 1 \)
\[ a_k = \frac{a_{k-1}}{k^2} = \frac{a_{k-2}}{k^2(k - 1)^2} = \cdots = \frac{a_0}{(k!)^2}. \] (28)
Setting the arbitrary constant \( a_0(r_1) = 1 \), the first solution is
\[ y_1(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} x^k = 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \ldots \] (29)

Second solution: Consider substituting
\[ y = y_1(x) \log x + \sum_{k=1}^{\infty} b_k x^k \] (30)
into (23). Then
\[ y' = \frac{y_1}{x} + y_1' \log x + \frac{d}{dx} \sum_{k=1}^{\infty} b_k x^k \] (31)
and
\[ xy'' = x \left[ -\frac{y_1}{x^2} + 2 \frac{y_1'}{x} + y_1'' \log x + \frac{d^2}{dx^2} \sum_{k=1}^{\infty} b_k x^k \right] \] (32)
so making cancellations we obtain
\[ L[y_1] \log x + 2 y_1' + L \left[ \sum_{k=1}^{\infty} b_k x^k \right] = 0. \] (33)
Now we know $L[y_1] = 0$ so this gives

$$L \left[ \sum_{k=1}^{\infty} b_k x^k \right] = -2y'_1$$

(34)

or in detail after appropriate index shifts to the first and second series

$$b_1 + \sum_{k=1}^{\infty} [(k + 1)k b_{k+1} + (k + 1)b_{k+1} - b_k] x^k = -2 - x - \frac{x^2}{6} - ...$$

(35)

Equating coefficients gives

$$b_1 = -2$$
$$4b_2 - b_1 = -1$$
$$9b_3 - b_2 = -\frac{1}{6}$$

\[\vdots\]

so

$$b_1 = -2, \quad b_2 = -\frac{3}{4}, \quad b_3 = -\frac{11}{108}, \quad ...$$

(36)

The second linearly independent solution is then

$$y_2(x) = y_1(x) \log x + \left[ -2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - ... \right]$$

(37)

**Example: Case 3 (log term required)**

Consider

$$Ly = xy'' + y = 0$$

(38)

with $p(x) = 0$ and $q(x) = x$ and a regular singular point at $x = 0$. The power series in $y_1$ and $y_2$ will converge for $|x| < \infty$ since $p$ and $q$ have convergent power series in this interval. The indicial equation is given by

$$r(r - 1) = 0$$

(39)

so $r_1 = 1$ and $r_2 = 0$.

**First solution:** Substituting $y = x^r \sum_{k=0}^{\infty} a_k x^k$ into (38) results in

$$\sum_{k=0}^{\infty} (r + k)(r + k - 1)a_k x^{r+k-1} + \sum_{k=0}^{\infty} a_k x^{r+k} = 0.$$ 

(40)

Shifting indices in the second series and regrouping terms gives

$$r(r - 1)a_0 x^{r-1} + \sum_{k=1}^{\infty} [(r + k)(r + k - 1)a_k + a_{k-1}] x^{r+k-1} = 0.$$ 

(41)

Setting the coefficient of $x^{r-1}$ to zero we recover the indicial equation with $r_1 = 1$ and $r_2 = 0$. Setting all the other coefficients to zero gives the recurrence relation

$$a_k = \frac{-a_{k-1}}{(r + k)(r + k - 1)}, \quad k \geq 1.$$ 

(42)

With $r = r_1$ this gives

$$a_k = \frac{-a_{k-1}}{(k + 1)k} = \frac{a_{k-2}}{(k + 1)k^2(k - 1)} = \ldots = \frac{(-1)^k a_0}{(k + 1)(k!)^2},$$

(43)
Setting the arbitrary constant \( a_0(r_1) = 1 \), the first solution is then

\[
y_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k)!} x^k = x - \frac{x^2}{2} + \frac{x^3}{12} - \frac{x^4}{144} \pm ... \tag{44}
\]

**Second solution:** First, let’s see how we run into trouble if we fail to include the log term in the second solution. The recurrence relation (42) with \( r = r_2 = 0 \) becomes (with \( b_k \) replacing \( a_k \) since we are now using \( r = r_2 \))

\[
b_k = -\frac{b_{k-1}}{k(k-1)}, \quad k \geq 1. \tag{45}
\]

This formula fails for \( k = 1 \). As was anticipated, for roots of the form \( r_2 - r_1 = N \) with \( N \in \mathbb{Z}^+ \) it may not be possible to determine \( b_N \) if the log term is omitted from \( y_2 \) (in our case \( N = 1 \)). For the second solution consider substituting

\[
y = cy_1(x) \log x + x^0 \sum_{k=0}^{\infty} b_k x^k \tag{46}
\]

into (38) so

\[
xy'' = \left[ -\frac{cy_1}{x} + 2cy_1' + cxy_1' \log x + x \frac{d^2}{dx^2} \sum_{k=0}^{\infty} b_k x^k \right]. \tag{47}
\]

We then obtain

\[
cL[y_1] \log x + 2cy_1' + \frac{cy_1}{x} + L \left[ \sum_{k=0}^{\infty} b_k x^k \right] = 0. \tag{48}
\]

Now we know \( L[y_1] = 0 \) so this gives

\[
L \left[ \sum_{k=0}^{\infty} b_k x^k \right] = -2cy_1' + \frac{cy_1}{x}. \tag{49}
\]

Expanding the left hand side gives

\[
b_0 + (2b_2 + b_1)x + (6b_3 + b_2)x^2 + (12b_4 + b_3)x^3 + (20b_5 + b_4)x^4 + ... \tag{50}
\]

and expanding the right hand side gives

\[
-c + \frac{3}{2} cx + \frac{5}{12} cx^2 + \frac{7}{144} cx^3 - \frac{1}{320} cx^4 \pm ... \tag{51}
\]

Equating coefficients gives the system of equations

\[
\begin{align*}
b_0 &= -c \\
2b_2 + b_1 &= \frac{3}{2} c \\
6b_3 + b_2 &= -\frac{5}{12} c \\
12b_4 + b_3 &= \frac{7}{144} c \\
&\vdots
\end{align*}
\]

Now \( b_0(r_2) \) is an arbitrary constant and \( c = -b_0 \). Notice that \( b_1 \) can also be chosen arbitrarily. This is because it is the coefficient of \( x^{r_1} = x^1 = x \) in the series

\[
x^{r_2} \sum_{k=0}^{\infty} b_k x^k = b_0 + x^{r_1}(b_1 + b_2 x + b_3 x^2 + \ldots). \tag{52}
\]
Consequently, modifying $b_1(r_2)$ changes subsequent coefficients $b_k(r_2)$ for $k > 1$ so as to effectively add a multiple of $y_1(x)$ to $y_2(x)$. In effect, changing $b_1$ just affects the choice of the arbitrary constant $a_0(r_1)$ that we already chose to be $a_0 = 1$. For convenience, we now choose $b_0 = 1$ and $b_1 = 0$. Then $b_2 = -3/4$, $b_3 = 7/36$, $b_4 = -35/1728$, ... so the second solution is

$$y_2(x) = -y_1(x) \log x + \left[ 1 - \frac{3}{4} x^2 + \frac{7}{36} x^3 - \frac{35}{1728} x^4 + \ldots \right]. \quad (53)$$

**Example: Case 3 (log term drops out)**

Consider

$$Ly = x^2 y'' + xy' + \left( x^2 - \frac{1}{4} \right) y = 0 \quad (54)$$

with $p(x) = 1$ and $q(x) = (x^2 - \frac{1}{4})$ and a regular singular point at $x = 0$. The power series in $y_1$ and $y_2$ will converge for $|x| < \infty$ since $p$ and $q$ have convergent power series in this interval. The indicial equation is given by

$$r(r-1) + r - \frac{1}{4} = 0 \quad (55)$$

so $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$.

**First solution:** Substituting $y = x^r \sum_{k=0}^{\infty} a_k x^k$ into (54) results in

$$\sum_{k=0}^{\infty} \left[ (r+k)(r+k-1) + (r+k) - \frac{1}{4} \right] a_k x^{r+k} + \sum_{k=0}^{\infty} a_k x^{r+k+2} = 0. \quad (56)$$

or shifting indices in the last series

$$\left( r^2 - \frac{1}{4} \right) a_0 x^r + \left[(r+1)^2 - \frac{1}{4}\right] a_1 x^{r+1} + \sum_{k=2}^{\infty} \left\{ \left[ (r+k)^2 - \frac{1}{4} \right] a_k + a_{k-2} \right\} x^{r+k} = 0. \quad (57)$$

Setting the coefficient of $x^r$ to zero we recover the indicial equation. Setting the other coefficients to zero we find

$$\left[(r+1)^2 - \frac{1}{4}\right] a_1 = 0 \quad (58)$$

and the recurrence relation

$$\left[(r+k)^2 - \frac{1}{4}\right] a_k = -a_{k-2}, \quad k \geq 2. \quad (59)$$

For $r_1 = \frac{1}{2}$ we then have

$$a_k = 0, \quad k = 1, 3, 5... \quad (60)$$

and

$$a_k = -\frac{a_{k-2}}{(k+1)^2}, \quad k = 2, 4, 6, ... \quad (61)$$

Then $a_2 = -a_0/3!, \; a_4 = a_0/5!, \; ...$ so letting $k = 2m$ in general

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \quad k = 1, 2, 3... \quad (62)$$

Choosing $a_0 = 1$, the first solution is

$$y_1(x) = x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m}. \quad (63)$$

**Second solution:** For the second solution in general we require a solution of the form

$$y = c y_1(x) \log x + x^{-1/2} \sum_{k=0}^{\infty} b_k x^k. \quad (64)$$
If the roots are of the form $r_2 - r_1 = N$ with $N \in \mathbb{Z}^+$ then the log term is generally required to enable the calculation of $b_N(r_2)$. However, in the present case, we see from (58) that the coefficient of $x^{-1/2}$ will vanish for $r_2 = -1/2$ regardless of the value of $b_1$ (where $b_1$ replaces $a_1$ since we are now using $r = r_2$). Hence, the log term is unnecessary in $y_2$ and $c = 0$. Let’s suppose we did not notice this ahead of time and attempted to substitute the general form (64) into (54). Then notice

$$y' = \frac{cy_1}{x} + cy_1' \log x + \frac{d}{dx} \left[ x^{-1/2} \sum_{k=0}^{\infty} b_k x^k \right]$$

(65)

and

$$y'' = -\frac{cy_1}{x^2} + 2 \frac{cy_1'}{x} + cy_1'' \log x + \frac{d^2}{dx^2} \left[ x^{-1/2} \sum_{k=0}^{\infty} b_k x^k \right]$$

(66)

We then obtain

$$c L[y_1] \log x + 2 c x y_1' + L \left[ x^{-1/2} \sum_{k=0}^{\infty} b_k x^k \right] = 0.$$  

(67)

Now we know $L[y_1] = 0$ so this gives

$$L \left[ x^{-1/2} \sum_{k=0}^{\infty} b_k x^k \right] = -2 c x y_1'.$$

(68)

Expanding the left hand side (for convenience we may use (58) and (59) with $b_k$ replacing $a_k$ since we are using $r = r_2$) gives

$$0 \cdot b_0 x^{-1/2} + 0 \cdot b_1 x^{1/2} + (2b_2 + b_0) x^{3/2} + (6b_3 + b_1) x^{5/2} + ...$$

(69)

and expanding the right hand side gives

$$-c x^{1/2} + \frac{5}{6} c x^{5/2} - \frac{9}{120} c x^{9/2} + ...$$

(70)

Equating coefficients give the system of equations

$$0 \cdot b_1 = -c$$

$$2b_2 + b_0 = 0$$

$$6b_3 + b_1 = \frac{5}{6} c$$

$$12b_4 + b_2 = 0$$

$$:.$$  

So $b_0(r_2)$ and $b_1(r_2)$ are both arbitrary as expected and $c = 0$. The other coefficients are then given by

$$b_{2k} = \frac{(-1)^k b_0}{(2k)!}, \quad b_{2k+1} = \frac{(-1)^k b_1}{(2k + 1)!}, \quad k = 1, 2, ...$$

(71)

Hence the second solution has the form

$$y_2(x) = x^{-1/2} \left[ b_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + b_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1} \right].$$

(72)

Note that as expected, $b_1$ just introduces a multiple of $y_1(x)$ so we may choose $b_1 = 0$. Setting the arbitrary constant $b_0 = 1$, the second solution finally becomes

$$y_2(x) = x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$  

(73)