Verify that it is not necessary to properly symmetrize a "bra" to calculate an inner product with a ket, for Fermions.

\[ \langle \beta_{u_1} \cdots \beta_{u_l} | \alpha_{1} \cdots \alpha_{n} \rangle = \prod_{l}! \sum_{\sigma} \langle \beta_{\sigma(1)} \cdots \beta_{\sigma(l)} | \alpha_{\sigma(1)} \cdots \alpha_{\sigma(n)} \rangle \]

\[ \text{symmetrized state.} \]

**Proof:**

\[ \langle \beta_{u_1} \cdots \beta_{u_l} | \alpha_{1} \cdots \alpha_{n} \rangle = \left( \frac{1}{m!} \sum_{\mu} \epsilon_{\mu} \beta_{\mu} \right) \langle \beta_{\mu(1)} \cdots \beta_{\mu(n)} | \alpha_1 \cdots \alpha_n \rangle \]

\[ \text{PHYSICAL KETS!} \]

\[ = \frac{1}{m!} \sum_{\mu} \sum_{\nu} \epsilon_{\nu} \beta_{\mu} \beta_{\nu} \langle \beta_{\nu(1)} \cdots \beta_{\nu(n)} | \alpha_1 \cdots \alpha_n \rangle \]

\[ = \frac{1}{m!} \sum_{\mu} \sum_{\nu} \epsilon_{\nu} \beta_{\mu} \beta_{\nu} \langle \beta_{\nu(1)} \beta_{\nu(2)} \cdots \beta_{\nu(n)} | \alpha_1 \cdots \alpha_n \rangle \]

\[ \text{Now:} \]

\[ \langle \beta_{u_1} \cdots \beta_{u_l} | \alpha_{1} \cdots \alpha_{n} \rangle = \frac{1}{m!} \sum_{\mu} \sum_{\nu} \epsilon_{\nu} \beta_{\mu} \beta_{\nu} \langle \beta_{\nu(1)} \beta_{\nu(2)} \cdots \beta_{\nu(n)} | \alpha_1 \cdots \alpha_n \rangle \]

\[ = \frac{1}{m!} \sum_{\mu} \sum_{\nu} \epsilon_{\nu} \beta_{\mu} \beta_{\nu} \langle \beta_{\nu(1)} \beta_{\nu(2)} \cdots \beta_{\nu(n)} | \alpha_1 \cdots \alpha_n \rangle \]

\[ \text{w/ use: } \epsilon_{\nu} = \epsilon_{\mu} \text{ and } \sum_{\nu} \beta_{\nu} = \beta_{\nu} \]

\[ \text{Perform a group} \]

\[ \text{ie } \beta_{\mu} \beta_{\nu} = \beta_{\nu} \beta_{\mu} \in G. \]

\[ \text{Collapse one of the sums} \]

\[ \text{at } \beta_{\mu} \text{ or } \beta_{\nu}. \]

\[ \Rightarrow \text{produce } m! \text{ terms.} \]

\[ \text{(possible permutations)} \]

\[ = \frac{1}{m!} \sum_{\nu} \sum_{\mu} \epsilon_{\nu} \beta_{\mu} \langle \beta_{\nu(1)} \beta_{\nu(2)} \cdots \beta_{\nu(n)} | \alpha_1 \cdots \alpha_n \rangle \]

\[ = \frac{1}{m!} \sum_{\nu} \sum_{\mu} \epsilon_{\nu} \beta_{\mu} \langle \beta_{\nu(1)} \beta_{\nu(2)} \cdots \beta_{\nu(n)} | \alpha_1 \cdots \alpha_n \rangle \]

\[ \text{Possible states} \]

\[ = \langle \beta_{u_1} \cdots \beta_{u_l} | \alpha_{1} \cdots \alpha_{n} \rangle \]

\[ \text{PHYSICAL STATE.} \]

\[ \text{ie no separate permutation symmetrization required for state } \langle \beta_{u_1} \cdots \beta_{u_l} | \alpha_{1} \cdots \alpha_{n} \rangle. \]
Question No. 3: Position probability densities of two identical particles

Basis: \( |1\Phi, 1\chi \rangle \in \mathcal{E}_r \) (Orbital state space)

Spin Basis \( |1\uparrow, 1\downarrow \rangle \in \mathcal{E}_s \) \( s_z(\pm) = \pm \frac{1}{2} + i \pm \frac{1}{2} \).

(a) \( \hat{S}_{\text{II}}(r, r') dr dr' \) - probability of finding one particle at \( r \) and one at \( r' \).

\( \hat{S}_I(r) dr \) - probability of finding one particle at \( r \).

→ How does \( \hat{S}_I \wedge \hat{S}_{\text{II}} \) look for a state, where one of the electrons occupies \( |1\Phi, +\rangle \) and one of \( |1\chi, +\rangle \)?

Solution 1: (Starting with Part B).

First apply Symmetrization postulate to the state \( |1\Phi, +, 2: \chi, +\rangle \) in Cohen-Tann. notation or as we noted in class (Jeynman notation) \( |1\Phi, +, \chi\rangle \).

The particles are fermions (electrons) and as such we apply the anti-symmetrizer:

\[
\hat{A} = \frac{1}{\sqrt{2}} \left( \hat{E}_2 \hat{P}_2 \right)
\]

\( \hat{P}_2 \): Permutation Operator

In the case of two particles this is very simple:

\[
A |1\Phi, +, \chi\rangle = \frac{1}{\sqrt{2}} \left( |1\Phi, +, \chi\rangle - |1\chi, +, \Phi\rangle \right) = \frac{1}{\sqrt{2}} \left( 1 - \hat{P}_2 \right) |1\Phi, +, \chi\rangle
\]

Now, evaluate the probability distribution that a particle is at position \( r \) and a second one is at \( r' \). Hence we have to project onto the corresponding state-vectors which are:

\[
|1\Phi, +, 1\chi\rangle \quad \text{unphysical state}
\]

Physical state:

\[
A |1\Phi^+, 1\chi^+, \rangle = \frac{1}{\sqrt{12}} \left( |1\Phi^+, 1\chi^+, \rangle - |1\chi^+, 1\Phi^+, \rangle \right) = \frac{1}{\sqrt{12}} \left( 1 - \hat{P}_2 \right) |1\Phi^+, 1\chi^+, \rangle
\]

The projection is: (Probability amplitude)

\[
\langle r, + | A |1\Phi^+, 1\chi^+, \rangle = \langle r, + | 1 - \frac{1}{2} (1 - \hat{P}_2)(1 + \hat{P}_2) |1\Phi^+, 1\chi^+, \rangle
\]

and:

\[
\frac{1}{2} (1 - \hat{P}_2)(1 + \hat{P}_2) = \frac{1}{2} (1 - \hat{P}_2^+ \hat{P}_2 + \hat{P}_2^+ \hat{P}_2) = (1 - \hat{P}_2) = (1 - \hat{P}_2)
\]

\( \ast \) since \( \hat{P}_2^+ = \hat{P}_2 \) (hermitian)
The projectors can be simplified, and we obtain

\[ A = \langle \gamma^+ \gamma^+ \rangle (1 - P_2) \langle 1 \gamma^+ \gamma^+ \rangle \]

\[ = \langle \gamma^+ \phi^+ \phi^+ \gamma^+ \rangle - \langle \gamma^+ \gamma^+ \gamma^+ \phi^+ \rangle = \phi \gamma^+ \phi^+ - \phi \gamma^+ \phi^+ \]

The probability density is given by the magnitude, i.e., \[ |A| \]

\[ |A|^2 = \langle A A^* \rangle = \langle \phi(r) \phi(r') \phi(r') \phi(r) \rangle = \frac{\pi}{8 \chi(r,r')} \]

The probability \[ s(r,r') \] is given by integrating over all possible positions \[ r' \]

\[ s(r,r') = \int \frac{\pi}{8 \chi(r,r')} \frac{d^3r'}{d^3r} \]

\[ = \int \frac{\pi}{8 \chi(r,r')} \frac{|\phi(r) \phi(r')|^2 d^3r'}{d^3r} \]

\[ = \int \frac{\pi}{8 \chi(r,r')} \frac{|\phi(r) \phi(r')|^2 d^3r'}{d^3r} \]

we assume that the states are orthogonal in \[ \mathbf{E} \mathbf{r} \gamma^+ \gamma^+ \phi^+ \phi^+ \gamma^+ \]

\[ \langle \phi | \gamma^+ \phi^+ \rangle \]

\[ = \langle \phi | \gamma^+ \phi^+ \rangle \]

\[ = 0 \]

\[ s_0 = |\phi(r)|^2 + |x(r)|^2 \]

- what happens if the states \[ (\gamma^+) \] and \[ (\phi) \] are not orthogonal in \[ \mathbf{E} \mathbf{r} \gamma^+ \gamma^+ \phi^+ \phi^+ \gamma^+ \]

- The integral over all space are:

\[ \int s(r,r') \frac{d^3r'}{d^3r} = \int |\phi(r)|^2 + |x(r)|^2 \]

\[ \int s_0(r,r') \frac{d^3r'}{d^3r} = \int |\phi(r)|^2 + |x(r)|^2 \]

\[ \text{(Number of particles)} \quad \text{Here it is not one!} \]

\[ \sum_{i=1}^{N} |\psi_i(r)|^2 < \psi_i ... \psi_i | = N \]
If we had used the symmetrizer approach our states and projector would be:

\[ |\Psi_{\text{initial}}\rangle = A^\dagger |\Psi + \chi\rangle = (1 - P_{12}) |\Psi + \chi\rangle \]

and the projector is:

\[ A = (1 - P_{12}) |\gamma_1, \gamma_2\rangle \]

Similarly to part B we had for:

\[ S(r) = |\langle r | \Psi + \chi \rangle|^2 \]

\[ = |\langle r_1, r_2 | \Psi + \chi \rangle|^2 \]

\[ = |\langle r_1, r_2 | (1 - P_{12}) |\Psi + \chi\rangle|^2 \]

\[ = |\langle r_1, r_2 | \Psi\rangle|^2 \]

Now since \[ <\gamma_1 | \gamma_2> = 0 \] (These two states are orthogonal states in the spin basis) since:

\[ <\gamma_1 | \gamma_2> = 1 \to 1 \otimes 1 \to 2 \Rightarrow <\gamma_1 | \gamma_2> = <1 \to 1 | 0 \otimes 1 \to 2 > = <1 | 1 > \otimes <1 | 2 > = 0 \]

we see that regardless of orthogonality of the states \[ \gamma \otimes \alpha \] in the \( r \) representation we obtain:

\[ |\langle r_1, r_2 | \Psi + \chi\rangle|^2 = |\langle r_1, r_2 | \Psi\rangle|^2 \]

\[ = |\langle r_1, r_2 | \Psi\rangle|^2 + |\langle r_1, r_2 | \alpha\rangle|^2 \]

\[ = \text{we have obtained the same result as before.} \]
Guestion No3A

Part A:

Since the particles are distinguishable (they have distinct spin and are such they live in different state spaces characterized by the "spinor representation" i.e.

\[ |\uparrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ |\downarrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \Rightarrow |\uparrow\rangle \otimes |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad \text{even if } |\uparrow\rangle \text{ and } |\downarrow\rangle \text{ are not orthogonal in } \Gamma \text{-representation.} \]

We do not need to apply the symmetrization theorem. However, if we do apply it, we get the same result as without (it would be detrimental if not!)

The initial state is: "|\uparrow\rangle\uparrow\rangle".

To find the probability of a electron bej at r or r' we note that the "unphysical" state associated with this is:

"|\Gamma, r\rangle" where the spin coordinate is suppressed.

\[ \text{ISTWAY} \]

* We can calculate \( s^{\Gamma}(r, r') \sim s^{\Pi}(r, r') \) directly by noting that:

- Both projection onto \( |\Gamma, r\rangle \quad \text{and} \quad |\Gamma, r'\rangle \) give us an electron at \( r \) or \( r' \).

Hence:

\[ s^{\Pi}(r, r') = |\langle r | r' + 1 \rangle_{\uparrow\rangle} |^2 + |\langle r | r' - 1 \rangle_{\downarrow\rangle} |^2 \]

\[ s^{\Gamma}(r, r') = |\langle r | r \rangle_{\uparrow\rangle} |^2 + |\langle r | r \rangle_{\downarrow\rangle} |^2 \]

We find \( s^{\Gamma}(r) \) by integrating over \( r \) or \( r' \) over all space \( d^3r \) and using the fact that \( \int s^{\Pi}(r) d^3r = \int s^{\Gamma}(r) d^3r = 1 \).

\[ s^{\Gamma}(r) = |\langle r | r \rangle_{\uparrow\rangle} |^2 + |\langle r | r \rangle_{\downarrow\rangle} |^2 \]

The results remain valid even for \( \langle 2| \uparrow\rangle \uparrow\rangle \) since states are orthogonal in \( \Gamma \).

Since we have not used the orthogonality in \( \Gamma \), this approach is only in \( \Gamma \).

\[ \text{NOTE: If we had used symmetries (A=\( \frac{1}{2\pi} \Gamma^2 \)\text{ex}) we could not have to keep track of which terms would contribute in the end (i.e. states |\Gamma, r\rangle \uparrow\rangle \quad \text{or} \quad |\Gamma, r\rangle \downarrow\rangle \downarrow\rangle \).} \]
Quantized single mode interacting with a two level system

Hamiltonian:
\[ H = \varepsilon_c |c\rangle \langle c| + \varepsilon_{12} |1\rangle \langle 1| + \varepsilon_{a} a^{+} a + \varepsilon_{b} b^{+} b + (K |c\rangle \langle c| a^{+} a) + (K |b\rangle \langle b| a^{+} a) \]

This Hamiltonian is familiar from 1st Term where we considered the two level system \(|1\rangle, |2\rangle \rangle\) basis.

When we reformulate the Hamiltonian using a field theory approach (and use operators \(b_a\) and \(b_a^\dagger\)) we obtain
\[ H = \varepsilon_c b_a^\dagger b_a + \varepsilon_b b_b^\dagger b_b + \varepsilon_a a^{+} a + \varepsilon_a b^\dagger a b^\dagger a + (K b_a^\dagger a b_a^\dagger a) \]

The equations of motion for \(b_a^\dagger b_a\) and \(a\), \(a^\dagger\) follow from straight-forward commutator algebra:

1. \[ [a^\dagger a, a] = a^\dagger a a - a a^\dagger a = a a^\dagger a - (1 - a a^\dagger) a = -a \]

2. \[ [a^\dagger a^\dagger a, a^\dagger a] = a^\dagger a a^\dagger a - a^\dagger a^\dagger a a^\dagger a = a a^\dagger a - a^\dagger a^\dagger a (a a^\dagger a - 1) a = +a^\dagger a \]

Now the Fermi-operators: call:
\[ \begin{align*}
   n_1 &= b_1^\dagger b_1 \\
   n_2 &= b_2^\dagger b_2
\end{align*} \]

\[ [b_1^\dagger b_1, b_2^\dagger b_2] = b_1^\dagger b_1 b_2 + b_2^\dagger b_2 b_1^\dagger b_1 \]

\[ = b_1^\dagger b_1 b_2 - (-\langle x | x \rangle) b_1^\dagger b_1 b_2 = -b_1^\dagger b_1 b_2 + b_1^\dagger b_1 b_2 \]

\[ = b_1^\dagger (-b_1^\dagger b_1 + 1) b_2 = b_1^\dagger b_1 b_2 \]

\[ = -b_1^\dagger b_1 b_2 \]

\[ \Rightarrow b_1^\dagger b_1 = -b_1 b_1^\dagger \]

\[ b_1^\dagger b_1 = b_2^\dagger b_2 \]

\[ = b_1^\dagger b_2 \]

\[ = b_1^\dagger b_2 \]
Commutators (continued)

(IV) \[ [b_1^+, b_1, b_2^+ b_1] = b_1^+ b_1 b_2^+ b_1 - b_2^+ b_1 b_1^+ b_1 \]
\[ = b_2^+ b_1^+ b_1 b_1 - b_2^+ b_1 b_1^+ b_1 \]
\[ = b_2^+ b_1^+ b_1 b_1 - b_2^+ (1-b_1^+ b_1) b_1 \]
\[ = -b_2^+ b_1 + b_2^+ b_1^+ b_1 \] and \[ b_1^+ b_1 = c \]
\[ = b_2^+ b_1 \]

(V) \[ [b_2^+ b_2, b_1^+ b_2] = b_2^+ b_2 b_2^+ b_2 - b_2^+ b_2 b_2^+ b_2 \]
\[ = -b_2^+ b_2 b_2^+ b_2 + b_2 b_2^+ b_2 b_2^+ \]
\[ = -b_2^+ b_2 b_2^+ b_2 + (1-b_2^+ b_2) b_2 b_2^+ \]
\[ = (-b_2^+ b_2 b_2^+ b_2 - b_2^+ b_2 b_2^+ b_2^+ b_2^+ b_2^+) + b_2 b_2^+ \]
\[ = +b_2 b_2^+ = -b_1^+ b_2 \]

(Vi) \[ [b_2^+ b_2, b_2^+ b_2^+] = (b_2^+ b_2 b_2^+ b_2^+ - b_2^+ b_2^+ b_2^+ b_2^+ b_2^+) \]
\[ = b_2^+ b_2 b_2^+ b_2 - b_2^+ b_2^+ b_2 b_2^+ \]
\[ = b_2^+ b_2 b_2^+ b_2 - b_2^+ (1-b_2^+ b_2^+) b_2 \]
\[ = -b_2^+ b_2 + b_2^+ b_2^+ b_2^+ b_2 + b_2 b_2^+ b_2^+ b_2 \]
\[ = -b_2^+ b_2 + b_2^+ (1-b_2^+ b_2) b_2 + b_2^+ (1-b_2^+ b_2^+) b_2 \]
\[ = -b_2^+ b_2 + 2b_2^+ b_2 + [b_2^+ b_2^+ b_2 b_2 + b_2^+ b_2 b_2 b_2] \]
\[ = 0 \quad \text{as babc (Pauli-Exc)} \]
\[ = +b_2 b_2^+ \]

Summarizing: \[ (b_c = b_2 \land b_r = b_1) \]

| \[ [b_c^+ b_c, b_r^+ b_c] = -b_r^+ b_c \] | Also we know: \[ [n_c, n_c] = 0 \] |
| \[ [b_c^+ b_c, b_c^+ b_r] = +b_c^+ b_v \] | \[ [b_{rV}, n_v] = 0 \] |
| \[ [b_v^+ b_r, b_r^+ b_c] = +b_{e_v} b_c \] | \[ [u_c, n_v] = 0 \] |
| \[ [b_v^+ b_{rV}, b_c^+ b_v] = -b_c^+ b_v \] | |

and: \[ [b_v^+ b_c, b_c^+ b_v] = -b_c^+ b_v + b_v^+ b_c \]
Correspondingly the equation of motion are:

1. For $a^+_t - a_t$ we obtain

$$\frac{d}{dt} a^+_t = \frac{i}{\hbar} \left[ H, a^+_t \right] = -i \frac{e}{\hbar} a^+_t + \frac{p}{\hbar} k \cdot b^+_t b_v$$
$$\frac{d}{dt} a^+_t = + \frac{e}{\hbar} a^+_t + \frac{p}{\hbar} k \cdot b^+_t b_c$$

2. Equation of motion for coherences $b^+_t b_c^* - b^+_t b_v$

$$\frac{d}{dt} \left( b^+_t b_c^* \right) = \frac{i}{\hbar} \left[ e_c \left[ b^+_t b_c, b^+_v b_c \right] + \frac{i}{\hbar} e_v \left[ b^+_v b_c, b^+_t b_c \right] + \frac{i}{\hbar} \left[ b^+_t b_c, b^+_v b_c \right] a^+ \right]$$
$$= \frac{i}{\hbar} \left( b^+_t b_c^* \right) - \frac{i}{\hbar} \left( b^+_t b_v^* \right)$$

$$= \frac{e}{\hbar} \left( e_c - e_v \right) b^+_t b_c^* + i \frac{p}{\hbar} \left( b^+_t b_v^* - b^+_t b_c^* \right) a^+$$

$$\frac{d}{dt} \left( b^+_t b_c^* \right) = - \frac{i}{\hbar} \left( e_c - e_v \right) b^+_t b_c^* + \frac{i}{\hbar} \left( n_c - n_v \right) a^+$$

Equations of motion for (interband) polarization.

Similarly:

$$\frac{d}{dt} \left( b^+_v b_c \right) = \frac{i}{\hbar} \left[ e_c \left[ b^+_v b_c, b^+_t b_v \right] + \frac{i}{\hbar} e_v \left[ b^+_t b_v, b^+_v b_c \right] + \frac{i}{\hbar} \left[ b^+_v b_c, b^+_t b_v \right] a^+ \right]$$

$$\frac{d}{dt} \left( b^+_v b_c \right) = + \frac{i}{\hbar} \left( e_c - e_v \right) b^+_v b_c^* + \frac{i}{\hbar} \left( n_c - n_v \right) a^+$$

$$\frac{d}{dt} \left( b^+_v b_c \right) = -i \frac{p}{\hbar} \left( b^+_v b_c^* \right)$$

$$\left( b^+_v b_c \right) = \left( b^+_t b_v \right)$$

3. Population: $\hat{n}_v$ and $\hat{n}_c$:

$$\frac{d}{dt} \left( b^+_v b_c^* \right) = \frac{i}{\hbar} \left[ e_v \left[ b^+_v b_c, b^+_v b_c \right] + \frac{i}{\hbar} e_c \left[ b^+_c b_v, b^+_v b_c \right] + \frac{i}{\hbar} \left[ b^+_v b_c, b^+_v b_c \right] a^+ \right]$$

$$\frac{d}{dt} \left( b^+_v b_c^* \right) = -i \frac{p}{\hbar} \left( b^+_v b_c^* \right)$$

Valence band population
similarity: \[ \frac{d}{dt} (bc^+bc) = + \frac{\alpha}{\hbar} \frac{\epsilon}{\hbar} \left[ a_g^+, b_{c^+bc} \right] + \frac{i}{\hbar} \frac{\epsilon}{\hbar} \frac{\epsilon^*}{\hbar} \left[ b_{c^+bc}, a_g^+ \right] \]

\[ \frac{d}{dt} (bc^+bc) = \frac{\alpha}{\hbar} \frac{\epsilon}{\hbar} \left( b_{c^+bc} - b_{c^+bc} \right) + \frac{i}{\hbar} \frac{\epsilon}{\hbar} \frac{\epsilon^*}{\hbar} \left( b_{c^+bc} + b_{c^+bc} \right) \]

As expected the populations are driven by the coherences\( b_{c^+bc} \)
and\( b_{c^+bc} \) as in the 1st term (compare results with\( P_{12}, P_{21}, P_1 \)
and\( P_2 \)).

Also we recover \[ \frac{d}{dt} (bc^+bc) = - \frac{d}{dt} (bc^+bc) \]
as in 1st Term.

**Comparison**

we see that the resulting equations of motion are really the
same as in the 1st Term, now having:

\[ \begin{align*}
P_{12} & \propto b_{c^+bc}^+ b_{c^+bc} \\
P_{21} & \propto b_{c^+bc} b_{c^+bc} \\
P_1 & \propto b_{c^+bc}^+ b_{c^+bc} \\
P_2 & \propto b_{c^+bc} b_{c^+bc}
\end{align*} \]

\[ \begin{align*}
\frac{d}{dt} P_{12} &= (\frac{d}{dt} P_{21})^* = i \frac{\epsilon}{\hbar} P_{12} + \frac{\epsilon^*}{\hbar} (P_{21} - P_1) \\
\frac{d}{dt} P_{21} &= -\frac{\epsilon}{\hbar} P_1 = \mu \frac{\epsilon^*}{\hbar} P_{12} + \mu \frac{\epsilon^*}{\hbar} P_{21}
\end{align*} \]

where we had the equations of motion:

\[ \frac{d}{dt} P_{12} = \frac{\epsilon}{\hbar} P_{21} = i \frac{\epsilon}{\hbar} P_{12} + \frac{\epsilon^*}{\hbar} (P_{21} - P_1) \]

\[ \frac{d}{dt} P_{21} = -\frac{\epsilon}{\hbar} P_1 = \mu \frac{\epsilon^*}{\hbar} P_{12} + \mu \frac{\epsilon^*}{\hbar} P_{21} \]

\[ i.e \text{ relaxation is driven by inversion } \propto P_1 - P_1; \]

\[ \text{Inversion is driven by relaxation.} \]

we see that in the quantum field approach the two operator
products (e.g.\( a_1^+ a_1 \),\( b_{c^+bc} \), etc.) yield identical equations of motion,
just like in 1st Term.