[4.1] Extend the time evolution analysis performed in class in which a quantized EM field and a two level system interact to account for the possibility of non-resonant interaction (i.e., the photon energy is not equal to the energy separation of the two level system). Find the system eigen-states and eigen-energies and compute the time evolution for the system if the electron is initially in the upper state.

[4.2] Use perturbation theory to estimate the corrections to state functions and to eigen-energies induced by coupling to the next nearest non-resonant states in problem [4.1].

[4.3] Consider the final result from the derivation begun in class:

\[
\dot{a} = -a(t) \int_{-\infty}^{t} dt' C_{\Delta}(t-t') \equiv -\left(\frac{\Gamma}{2} + i\delta\Omega\right) a(t)
\]

\[
C_{\Delta}(t-t') = \frac{1}{\hbar^2} \int dE' \left| \mu(E') \right|^2 \rho(E') \exp \left(-\frac{i\hbar}{\hbar}(E'-E)(t-t') \right)
\]

for coupling of a state with slowing varying amplitude \(a(t)\) to a continuum. Complete the derivation by finding the decay rate and frequency shift quantities defined above. In particular, show that they are given by the following expressions:

\[
\Gamma = \frac{\pi}{\hbar} \rho(E) \left| \mu(E) \right|^2
\]

\[
\delta\Omega = \frac{1}{\hbar} \text{PV} \int dE' \frac{\rho(E') \left| \mu(E') \right|^2}{E'-E}
\]

where PV means principal value. Hint: apply a convergence factor and evaluate the time integral as a limit.

[4.4] Consider a system consisting of a single quantized HO and bath consisting of a very large number of HOs. Let there exist a weak coupling between the system and the bath such that the Hamiltonian for bath and system has the form:

\[
H = H_{\text{System}} + H_{\text{Bath}} + W
\]

\[
H_{\text{System}} = \hbar \Omega a \dagger a
\]

\[
H_{\text{Bath}} = \sum_n \hbar \omega_n b_n \dagger b_n
\]

\[
W = \sum_n \left( \hbar \kappa_n b_n \dagger a + \hbar \kappa_n^* a \dagger b_n \right)
\]

where creation and destruction operators belonging to different HOs commute. By adapting the approach applied in class, show that the following damped equation of motion results (under appropriate conditions):

\[
\frac{d\alpha}{dt} = -\left(\frac{\Gamma}{2} + i\delta\Omega\right) \alpha + f(t)
\]
where the following definitions for slowly varying operators are used:

\[ a = \alpha e^{-i\Omega t} \]
\[ b_n = \beta_n e^{-i\omega_n t} \]

In addition to finding expressions for the damping and frequency shift terms, show that the forcing function \( f(t) \) is given by:

\[ f(t) = -\sum_n \kappa_n^* \beta_n(0) e^{-i(\omega_n - \Omega) t} \]

Suppose that \( f(t) \) is set to zero. Compute the commutator of the system raising and lowering operators in the presence of damping. Do you see a problem here?