a.) Derive equations of motion for P₁ and P₂.
b.) Use the expression for the steady state polarization amplitudes P₁₂ (and P₂₁) to derive the following inversion density rate equation:

\[
\frac{d\Delta N}{dt} = -2 \, c \, g(\Delta N, \omega) \, P - \frac{\Delta N - \Delta N}{T₁}
\]

where,

\[
\Delta N = \left( |b|^2 - |a|^2 \right)
\]

\[
P \equiv \frac{2 \varepsilon_0 E^2}{\hbar \omega} \equiv \text{Photon density}
\]

c.) Solve for \(\Delta N\) under steady state conditions to show that the steady state gain \(g_{ss}\) is given by,

\[
g_{ss}(P, \omega) = \frac{g(\Delta N, \omega)}{1 + \frac{P}{P_{Sat}}} + P_{Sat}
\]

where \(P_{Sat}\) is called the “saturation photon density” and is to be defined by you.

d.) Compute the saturation photon density \(P_{sat}\) at transition line center for a dilute two-level system having the following properties:

- Dipole matrix element of \((1 \text{ Angstrom}) \times q\) where \(q\) is the electron charge
- Transition energy of \(0.8 \text{ eV}\)
- \(T₂ = 100 \text{ fsec.}\)

Compute for \(T₁\) values of 1 nsec and 1 psec.

e.) Compute the unsaturated optical gain (i.e., gain with \(P=0\)) in units of dB/cm for the same medium at concentrations \(N=10^{16} \text{ cm}^{-3}\) and \(N=10^{18} \text{ cm}^{-3}\). For simplicity, assume that the two-level systems are fully inverted (i.e., \(\Delta N=\text{N}\)).

[2.2] The following equation for optical photon density is easily deduced for propagation in a medium with optical gain \(g_{ss}\) given by the expression derived in [2.1]

\[
\frac{dP}{dz} = g_{ss} \, P
\]
Show that a traveling-wave optical amplifier has a saturated gross gain $G$ given approximately by the expression:

$$G \equiv \frac{P_{\text{Out}}}{P_{\text{in}}} \approx G_{\text{Max}} e^{-P_{\text{Out}}/P_{\text{Sat}}}$$

where the gain medium saturates according to the expression derived in [2.1] and where the saturation is assumed only moderate so that $G \gg 1$ can be assumed to approximately solve the equation.

[2.3] Compute the phase change induced in the optical field by propagation through 1 cm of the medium in [2.1] at the same concentrations. For this calculation you will need to assume that the optical frequency is detuned from the transition line center (assume a detuning equal to two line widths of the transition). Could this effect be put to use to build a modulator of optical power?