For a two-level system, the time evolution can be written as:

\[
\begin{pmatrix}
    a(t) \exp \left(-\frac{i}{\hbar} E_1 t \right) \\
    b(t) \exp \left(-\frac{i}{\hbar} E_2 t \right)
\end{pmatrix} = \begin{pmatrix}
    E_1 & \tilde{\mu}_{12} \cdot \tilde{E}(\tilde{x}_0, t) \\
    \tilde{\mu}_{12} \cdot \tilde{E}(\tilde{x}_0, t) & E_2
\end{pmatrix} \begin{pmatrix}
    a(t) \exp \left(-\frac{i}{\hbar} E_1 t \right) \\
    b(t) \exp \left(-\frac{i}{\hbar} E_2 t \right)
\end{pmatrix}
\]

Simplify the equation \(\Rightarrow\)

\[
\begin{cases}
    a = -\frac{i}{\hbar} \tilde{\mu}_{12} \cdot \tilde{E}(\tilde{x}_0, t) \exp(-i\omega_{12} \cdot t) b \\
    b = -\frac{i}{\hbar} \tilde{\mu}_{21} \cdot \tilde{E}(\tilde{x}_0, t) \exp(i\omega_{12} \cdot t) a
\end{cases}
\]

where \(\omega_{12} = \frac{E_2 - E_1}{\hbar}\)

Replace \(\tilde{E}(\tilde{x}_0, t)\) with \(\frac{1}{2} \tilde{E}_0 (e^{i\omega t} + e^{-i\omega t})\), define Rabi frequency \(\Omega_R = \left| \frac{\tilde{\mu}_{12} \cdot \tilde{E}_0}{\hbar} \right|\) and consider rotating wave approximation, we have:

\[
\begin{cases}
    a = -\frac{i}{2} e^{i\delta\omega \cdot t} \Omega_R b \\
    b = -\frac{i}{2} e^{-i\delta\omega \cdot t} \Omega_R a
\end{cases}
\]

(1)

Take another time derivative of \(a\) and \(b\), \(\Rightarrow\)

\[
\begin{cases}
    a = -\frac{i}{2} \Omega_R \exp(i\delta\omega \cdot t) b + \frac{1}{2} \Omega_R \cdot \delta\omega \cdot \exp(i\delta\omega \cdot t) b \\
    b = -\frac{i}{2} \Omega_R \exp(-i\delta\omega \cdot t) a - \frac{1}{2} \Omega_R \cdot \delta\omega \cdot \exp(i\delta\omega \cdot t) a
\end{cases}
\]

(2)

From equation (1), we get:

\[
\begin{cases}
    a - i\delta\omega \cdot a + \frac{1}{4} \Omega_R^2 \cdot a = 0 \\
    b + i\delta\omega \cdot b + \frac{1}{4} \Omega_R^2 \cdot b = 0
\end{cases}
\]

Solve the ODE(Ordinary Differential Equation) above \(\Rightarrow\)

\[
\begin{align*}
    a(t) &= A_1 e^{\frac{\Omega^2}{4} t} + A_2 e^{\frac{\Omega^2}{4} t (i\beta - \delta\omega)} \\
    b(t) &= B_1 e^{\frac{\Omega^2}{4} t (i\beta + \delta\omega)} + B_2 e^{\frac{\Omega^2}{4} t (i\beta - \delta\omega)}
\end{align*}
\]

where \(\beta = \sqrt{\Omega_R^2 + \delta\omega^2}\)

If we assume that the electron is initially in the upper state, it means that
\[ a(0) = 0, \quad \Rightarrow b(0) = 0 \]

\[ b(0) = 1, \quad \Rightarrow a(0) = -\frac{i}{2} \Omega_R \]

From \( a(0) = 0 \) and \( a(0) = -\frac{i}{2} \Omega_R \), we can derive the expression for \( A_1 \) and \( A_2 \):

\[
\begin{cases}
  a(0) = 0 & \Rightarrow A_1 + A_2 = 0 \\
  a(0) = -\frac{i}{2} \Omega_R & \Rightarrow \frac{i}{2} A_1 (\delta \omega + \beta) + \frac{i}{2} A_2 (\delta \omega - \beta) = -\frac{i}{2} \Omega_R \\
  \Rightarrow A_1 = -A_2 = -\frac{\Omega_R}{2\beta}
\end{cases}
\]

Similarly, from \( b(0) = 1 \) and \( b(0) = 0 \):

\[ B_1 = \frac{\delta \omega + \beta}{2\beta} \quad \text{and} \quad B_2 = -\frac{\delta \omega + \beta}{2\beta} \]

So the time evolution of the upper and lower energy state amplitudes of the 2-level systems is:

\[
a(t) = A_1 e^{\frac{i}{2} t (\delta \omega + \beta)} + A_2 e^{\frac{i}{2} t (\delta \omega - \beta)} = -e^{\frac{i}{2} t \delta \omega} \frac{i}{2} \sin\left(\frac{t}{2} \Omega_R \sqrt{1 + \left(\frac{\delta \omega}{\Omega_R}\right)^2}\right)
\]

\[
b(t) = B_1 e^{\frac{i}{2} t (-\delta \omega + \beta)} + B_2 e^{\frac{i}{2} t (-\delta \omega - \beta)}
\]

\[
e^{-\frac{i}{2} t \delta \omega} \left[ \cos\left(\frac{t}{2} \Omega_R \sqrt{1 + \left(\frac{\delta \omega}{\Omega_R}\right)^2}\right) + i \frac{\delta \omega}{\Omega_R} \sin\left(\frac{t}{2} \Omega_R \sqrt{1 + \left(\frac{\delta \omega}{\Omega_R}\right)^2}\right) \right]
\]

The probabilities for finding an electron in the upper and lower states are given by:

\[
P_a = |a(t)|^2 = a(t) \cdot (a(t))^* = \frac{1}{1 + \left(\frac{\delta \omega}{\Omega_R}\right)^2} \sin^2\left(\frac{t}{2} \Omega_R \sqrt{1 + \left(\frac{\delta \omega}{\Omega_R}\right)^2}\right)
\]
Here are the plots of probabilities for finding an electron in the upper and lower state, with detuning value, $\frac{\delta \omega}{\Omega_R}$, of 0, 1, and 3, respectively.
Probabilities for finding an electron in the upper (Pb) and lower state (Pa): Detuning is 1

Probabilities for finding an electron in the upper (Pb) and lower state (Pa): Detuning is 3
\[ I = \frac{P}{A} = \frac{ce|E|^2}{2n} = \frac{ce_r\varepsilon_0|E|^2}{2n} \]

[1.2] \[ n = \sqrt{\varepsilon_r} \]
\[ \Rightarrow |E| = \sqrt{\frac{2P}{c\varepsilon_r\varepsilon_0A}} \]

The angular Rabi-oscillation frequency is given by:
\[ \Omega_R = \left| \frac{\mathbf{\mu} \cdot \vec{E}}{\hbar} \right| = \left| \frac{\mathbf{\mu} \cdot \vec{E}}{\hbar} \right| \]

where the dipole strength is $1.6 \times 10^{-28}$ (m-C)

| P     | Field (V/m ) strength= $\frac{2P}{c\varepsilon_r\varepsilon_0A}$ | $\Omega_R = \frac{|\mathbf{\mu} \cdot \vec{E}|}{\hbar}$ (Hz) |
|-------|-------------------------------------------------|-------------------------------------------------|
| 1mW   | $4.61\times10^2$                                | $7\times10^8$                                  |
| 100 mW| $4.61\times10^3$                                | $7\times10^9$                                  |
| 10W   | $4.61\times10^4$                                | $7\times10^{10}$                               |

[1.3]
Method I: From Maxwells equations in free space:
\[
\begin{align*}
\nabla \times \vec{B} &= \partial_t \vec{E} \\
\nabla \times \vec{E} &= -\partial_t \vec{B} \\
\n\nabla \cdot \vec{B} &= 0 \\
\n\nabla \cdot \vec{D} &= \rho \\
\n\Rightarrow \vec{B} &= \nabla \times \vec{A}, \text{ where } \vec{A} \text{ is a vector potential.} \\
\n\text{So } \nabla \times (\vec{E} + \partial_t \vec{A}) &= 0 \quad (1) \\
\n\text{Since gradient fields are conservative(stokes theorem), i.e. } \nabla \times (\nabla \phi) &= 0 \quad (2) \\
\n\text{From (1) and (2), } \\
\n\Rightarrow \vec{E} &= -\nabla \phi - \partial_t \vec{A}, \text{ where } \phi \text{ is a scalar potential} \\
\n\text{So electric and magnetic fields can be written in terms of scalar and vector potential, as follows:} \\
\n\begin{align*}
\vec{E} &= -\nabla \phi - \partial_t \vec{A} \\
\vec{B} &= \nabla \times \vec{A} \\
\end{align*} \quad (3) \\
\n\text{However, this prescription is not unique. Many different potentials can generate the same fields. This is called gauge invariance.} \\
\text{A possible transformation for the vector potential is:} \\
\vec{A}' &= \vec{A} - \nabla \psi
Because $\vec{A}'$ leaves the magnetic field $\vec{B}$ unchanged:

$$\Rightarrow \vec{B} = \nabla \times \vec{A}' = \nabla \times \vec{A} - \nabla \times \nabla \psi = \nabla \times \vec{A}; \quad (\nabla \times \nabla \psi = 0)$$

The electric field $\vec{E}$ is not changed, too:

$$\Rightarrow \vec{E} = -\nabla \phi' - \partial_i \vec{A}' = -\nabla \phi' - \partial_i (\vec{A} - \nabla \psi)$$

$$\Rightarrow \nabla \phi' = \nabla \phi - \partial_i \psi$$

So the following transformation leaves the $\vec{E}$ and $\vec{B}$ fields unchanged:

$$\begin{cases} 
\vec{A}' = \vec{A} - \nabla \psi \\
\phi' = \phi - \partial_i \psi 
\end{cases}$$

Method 1: Here we need to choose a gauge which results in cancellation of the vector term and creation of a corresponding scalar potential term of dipole form.

Assume a monochromatic plane wave:

$$\vec{E}(\vec{r},t) = E_0 \cos(kz - \omega t) \hat{x}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c} E_0 \cos(kz - \omega t) \hat{y}$$

$$\therefore \vec{B} = \nabla \times \vec{A} \Rightarrow \vec{A} = \frac{1}{ck} E_0 \sin(kz - \omega t) \hat{x}$$

Now we need vector term vanish, i.e. $\vec{A}' = 0$

$$\Rightarrow \vec{A} - \nabla \psi = 0 \Rightarrow \nabla \psi = \frac{1}{ck} E_0 \sin(kz - \omega t) \hat{x} \quad (4)$$

We can simplify the equation above by using Dipole Approximation, which says that the wavelength of the type of the electromagnetic wave interacting with the atomic system is much larger than the typical size of a atom, thus, $kz = 2\pi \frac{z}{\lambda} \to 0$

From equation (4) $\Rightarrow \psi = \frac{1}{ck} E_0 x \sin(\omega t)$

The scalar potential now is transformed to:

$$\phi' = \phi - \partial_i \psi = \phi - \partial_i \psi = \phi - E_0 x \cdot \cos(\omega t) \text{ (dipole form)}$$

So with the new gauge, the vector potential term vanishes, and a corresponding scalar potential term of dipole form is created.
Method II: Here we need to replace \( \langle l|\vec{p}|2\rangle \) with a term similar to \( \langle l|\vec{E} \cdot \vec{r}|2\rangle \).

i. First, consider the commutator \( [H_0, r] \) between eigenstates 1 and 2.

\[
\langle l|[H_0, r]|2\rangle = \langle l|H_0 r - rH_0|2\rangle = (E_1 - E_2) \langle l|r|2\rangle \tag{1}
\]

(Because \( H_0 \) is Hamiltonian so that \( H_0 = H_0^* \), i.e. \( \langle l|H = E_1\langle l| \)

ii. While there is another way to calculate commutator \( [H_0, r] \) between eigenstates 1 and 2:

\[
H_0 = \frac{p^2}{2m} + V(r) \Rightarrow
\]

\[
\langle l|[H_0, r]|2\rangle = \langle l|[\frac{p^2}{2m} + V(r), r]|2\rangle \Rightarrow
\]

\[
= \frac{1}{2m} \langle l|[p^2, r]|2\rangle = \langle l|p^2r - rp^2|2\rangle = \frac{1}{2m} \langle l|p^2r - prp + prp - rp^2|2\rangle \tag{2}
\]

\[
= \frac{1}{2m} \langle l|p[p, r] + [p, r]p|2\rangle \Rightarrow
\]

\[
= \frac{-2i\hbar}{2m} \langle l|p|2\rangle
\]

Compare (1) and (2) we have:

\[
\frac{-2i\hbar}{2m} \langle l|p|2\rangle = (E_1 - E_2) \langle l|r|2\rangle \Leftrightarrow \langle l|p|2\rangle = \frac{im}{\hbar} (E_1 - E_2) \langle l|r|2\rangle
\]

[1.4] If we assume there is only coupling between states 1 and 2, and that between states 2 and 3, the Hamiltonian for this three level system is:

\[
H = \begin{pmatrix}
E_1 & \tilde{\mu}_{12} \cdot \vec{E}(\vec{x}_0, t) & 0 \\
\tilde{\mu}_{21} \cdot \vec{E}(\vec{x}_0, t) & E_2 & \tilde{\mu}_{23} \cdot \vec{E}(\vec{x}_0, t) \\
0 & \tilde{\mu}_{32} \cdot \vec{E}(\vec{x}_0, t) & E_3
\end{pmatrix}
\]

Three eigenstates can be written w/ column vectors:

\[
|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \text{and} \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

7
The time evolution of the system can be written as:

$$\psi(t) = a(t)e^{-\frac{iE_1 t}{\hbar}}|1\rangle + b(t)e^{-\frac{iE_2 t}{\hbar}}|2\rangle + c(t)e^{-\frac{iE_3 t}{\hbar}}|3\rangle$$

Solve the Hamiltonian equation \(i\hbar \partial_t \psi(t) = H \psi(t)\), i.e.

$$\begin{pmatrix}
    a(t)e^{-\frac{iE_1 t}{\hbar}} \\
    b(t)e^{-\frac{iE_2 t}{\hbar}} \\
    c(t)e^{-\frac{iE_3 t}{\hbar}}
\end{pmatrix} = \begin{pmatrix}
    E_1 & \bar{\mu}_{12} \cdot \bar{E}(\bar{x}_0,t) & 0 \\
    \bar{\mu}_{21} \cdot \bar{E}(\bar{x}_0,t) & E_2 & \bar{\mu}_{23} \cdot \bar{E}(\bar{x}_0,t) \\
    0 & \bar{\mu}_{32} \cdot \bar{E}(\bar{x}_0,t) & E_3
\end{pmatrix} \begin{pmatrix}
    a(t)e^{-\frac{iE_1 t}{\hbar}} \\
    b(t)e^{-\frac{iE_2 t}{\hbar}} \\
    c(t)e^{-\frac{iE_3 t}{\hbar}}
\end{pmatrix}$$

Simplify the equation =>

$$\begin{align*}
    a &= -\frac{i}{\hbar} \bar{\mu}_{12} \cdot \bar{E}(\bar{x}_0,t)e^{-i\omega_{12} t}b \\
    b &= -\frac{i}{\hbar} \bar{\mu}_{21} \cdot \bar{E}(\bar{x}_0,t)e^{i\omega_{12} t}a - \frac{i}{\hbar} \bar{\mu}_{23} \cdot \bar{E}(\bar{x}_0,t)e^{-i\omega_{23} t}c \\
    c &= -\frac{i}{\hbar} \bar{\mu}_{32} \cdot \bar{E}(\bar{x}_0,t)e^{i\omega_{23} t}b
\end{align*}$$

(1)

where \(\omega_{12} = \frac{E_2 - E_1}{\hbar}\) and \(\omega_{23} = \frac{E_3 - E_2}{\hbar}\)

Replace \(\bar{E}(\bar{x}_0,t)\) with \(\frac{L}{2\hbar} \bar{E}_0 (e^{i\omega t} + e^{-i\omega t})\), define Rabi frequency \(\Omega_{R12} = \frac{\bar{\mu}_{12} \cdot \bar{E}_0}{\hbar}\),

\(\Omega_{R23} = \frac{\bar{\mu}_{23} \cdot \bar{E}_0}{\hbar}\) and consider rotating wave approximation (i.e. dropping the fast oscillation frequency term), we have:

$$\begin{align*}
    a &= -\frac{i}{2}e^{i\delta\omega_{12} t} \Omega_{R12} b \\
    b &= -\frac{i}{2}e^{-i\delta\omega_{12} t} \Omega_{R12} a - \frac{i}{2}e^{i\delta\omega_{23} t} \Omega_{R23} c \\
    c &= -\frac{i}{2}e^{-i\delta\omega_{23} t} \Omega_{R23} b
\end{align*}$$

where \(\delta\omega_{12} = \omega - \omega_{12}\) and \(\delta\omega_{23} = \omega - \omega_{23}\)

We can solve the equations above like we did in problem set 1, but it will involve a three order ODE, which is not trivial.

So we can make use of the unperturbed solution, which we have in solution to problem 1, to solve \(c(t)\).
For a two level system with initial state of $a(0) = 1$ and $b(0) = 0$

\[ b(t) = -\frac{i}{\sqrt{\Omega R_{12}} + \frac{(\delta \omega_{12})^2}{2}} e^{-\frac{t}{2} \cdot \delta \omega_{12}} \sin\left(\frac{t}{2} \Omega R_{12} \sqrt{1 + \left(\frac{\delta \omega_{12}}{\Omega R_{12}}\right)^2}\right) \]

\[ c = -\frac{i}{2} e^{-i \delta \omega_{23} t} \Omega R_{23} b \]

\[ c(t) = -\frac{i}{2} \int_{0}^{t} \Omega R_{23} b(t) dt \]

\[ c(t) = -\frac{i}{2} \frac{\Omega R_{23} \Omega R_{12}}{2 \delta \omega_{12}} \int e^{-\frac{i}{2} t \cdot (\delta \omega_{12} + 2 \delta \omega_{23})} \sin\left(\frac{t}{2} \delta \omega_{12}\right) dt \]

\[ c(t) = -\frac{\Omega R_{23} \Omega R_{12}}{4 i \delta \omega_{12}} \left[ e^{-\frac{i}{2} t \cdot (\delta \omega_{12} + 2 \delta \omega_{23} - \delta \omega_{12}')} - e^{-\frac{i}{2} t \cdot (\delta \omega_{12} + 2 \delta \omega_{23} + \delta \omega_{12}')} \right] dt \]

\[ = -\frac{\Omega R_{23} \Omega R_{12}}{4 i \delta \omega_{12}} \left\{ \frac{-\frac{i}{2} t \cdot (\delta \omega_{12} + 2 \delta \omega_{23} - \delta \omega_{12}')}{e} - \frac{-\frac{i}{2} t \cdot (\delta \omega_{12} + 2 \delta \omega_{23} + \delta \omega_{12}')} {e} \right\} + C_{0} \]

\[ = C_{1}(t) + C_{0} \]

where $C_{0}$ is a constant, and
\[ C_1(t) = -\frac{\Omega_{R23}^2 \Omega_{R12}}{4i \delta \omega_{12}} \left\{ \begin{array}{c} \frac{-i}{2}t(\delta \omega_{12} + 2\delta \omega_{23} - \delta \omega'_{12}) + \frac{i}{2} \delta \omega'_{12} \\ \frac{i}{2}t(\delta \omega_{12} + 2\delta \omega_{23} - \delta \omega'_{12}) - \frac{i}{2} \delta \omega'_{12} \\ \end{array} \right\} + \frac{\delta \omega'_{12}}{(\delta \omega_{12} + 2\delta \omega_{23})^2 - \delta \omega'_{12}^2} \left\{ \begin{array}{c} \cos\left(\frac{1}{2}t \cdot \delta \omega'_{12}\right) \\ \sin\left(\frac{1}{2}t \cdot \delta \omega'_{12}\right) \\ \end{array} \right\} \]

and according to initial condition, where \( c(0) = 0 \Rightarrow \)

\[ C_0 = -\frac{\Omega_{R23}^2 \Omega_{R12}}{(\delta \omega_{12} + 2\delta \omega_{23})^2 - \delta \omega'_{12}^2} \]

From calculation above, we know that \( c(t) \) is a function with period of \( \frac{4\pi}{\delta \omega_{12}} \).

In addition,

\[ |c(t)|^2 \propto \cos(t \cdot \delta \omega'_{12}), \] which means that the \( c(t) \) is maximized at \( t = \frac{2\pi}{\delta \omega'_{12}} \)

\[ |c(\frac{2\pi}{\delta \omega_{12}})|^2 = |C_1(\frac{2\pi}{\delta \omega_{12}}) + C_0|^2 = \left| -\frac{2\Omega_{R23}^2 \Omega_{R12}}{(\delta \omega_{12} + 2\delta \omega_{23})^2 - \delta \omega'_{12}^2} \right|^2 \]

* Here we assume that the Rabi frequency is a tiny fraction of the detuning frequency between the electric field and the 3-2 transition frequency, i.e.

\[ \Omega_{R23}, \Omega_{R12} \ll \delta \omega_{23} \]

** Furthermore, the electric field is nearly resonant with the 1-2 transition \Rightarrow \]

\[ \delta \omega_{12} \ll \delta \omega_{23} \]
The amplitude of $c(t)$ is very small so it’s valid that we consider just the leading order term.