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Projective Transformations of Color-Mixture Diagrams*

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Projective transformations of a standard coordinate system can be written in the form:

\[ u = c_1 + (c_2+c_3)/c_4 + (c_2+c_3)/c_4 + 1 \]
\[ v = c_1 + (c_2+c_3)/c_4 + (c_2+c_3)/c_4 + 1 \]

The shapes and relative sizes of all geometrical figures subject to such transformations will be invariant if the values of \( c_1 \) and \( c_4 \) are invariant, and if the following quantities are also invariant:

\[ (c_2 - c_3)(c_2 + c_3) \]
\[ (c_2 + c_3)(c_2 + c_3) \]

The sizes of such figures are invariant when the numerators and denominators of these expressions are themselves invariant.

Colors having equally noticeable differences can be represented by equidistant points in some projective transformation of the standard coordinate system. In the standard diagram the separations of the points on any line representing a series of equally noticeably different colors are proportional to the square of the distance from some unique point on that line, and if the loci of such unique points for all linear series of colors is a straight line. The locus of colors equally noticeably different from any standard color can be represented adequately in the standard color-mixture diagram by an ellipse. All such ellipses can be transformed by a single projective transformation into equal-sized circles only if the common tangents of every pair of ellipses intersect on a single straight line. This line is the same as the locus of unique points for the series of colors just described. The equation of this straight line is \( (c_2 + c_3 + c_4 + 1 = 0) \), where the constants are those appearing in the denominators of the successful transformation formulas. If the separations of points representing equally noticeably different colors in the standard color-mixture diagram cannot be represented adequately in the manners described, then such colors cannot be represented adequately by equidistant points in any projective transformation of the standard coordinate system.

The interest in the projective transformation of color-mixture diagrams has increased recently as a consequence of attempts to find transformations in which all pairs of colors having equally noticeable differences of color can be represented by pairs of equidistant points. In a previous publication the author pointed out some influence of the various coefficients of the transformation equations on the shape of the spectrum locus in the resulting diagrams. D. B. Judd has requested this further development of the conditions which must be fulfilled if two transformations are to result in spectrum loci having identical shapes.

Neither the present paper nor the previous publication^ should be interpreted as evidence of agreement with suggestions that the standard color-mixture diagram be abandoned in favor of some transformed diagram exhibiting approximately uniform spacing of equally noticeable color differences. The author considers the present ICI standard coordinate system^ for colorimetry adequate for the specification of color and color tolerances, and believes that the confusion which would arise from its abandonment after more than ten years of extensive use would seriously discredit the science of color.

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The rectangular coordinates of the point representing a color in the standard color-mixture diagram will be designated by \( x \) and \( y \). The rectangular coordinates of the corresponding point in a projective transformation of the standard color-mixture diagram will be designated by \( u \) and \( v \). The equations connecting \( u \) and \( v \) with \( x \) and \( y \) may be written in the form
\[
\begin{align*}
u &= (e_2x + e_3y + e_4)/(e_1x + e_3y + 1), \quad (1) \\
v &= (e_2x + e_3y + e_4)/(e_1x + e_3y + 1). \quad (2)
\end{align*}
\]

The coefficients, \( e_1 \cdots e_8 \), can be assigned any values. The dependence of the transformed shape of the color-mixture diagram on these coefficients has been described elsewhere.* In particular, from a geometrical interpretation of \( e_1 \) and \( e_3 \) given on pages 296 and 297 of the previous paper, it is evident that all sets of transformation equations which yield any particular shape of spectrum locus must have identical denominators. This same conclusion can also be drawn from the fact that, in the spatial interpretation of the projective transformation, the line \( (e_2x + e_3y + 1 = 0) \) is the intersection of the plane \( (x, y) \) with the plane passing through the center of projection, \( P \), and parallel to the plane \( (u, v) \). Since another projection of the spectrum locus from the center, \( P \), can yield the same shape of spectrum locus only if the plane of the new projection is parallel to the plane \( (u, v) \), the plane through \( P \) and parallel to the new plane of projection must remain unchanged and therefore also the line of intersection \( (e_2x + e_3y + 1 = 0) \), which defines the denominator of the modified set of transformation equations. Since a change of the center of projection, \( P' \), can change the shape of the spectrum locus projected onto a plane parallel to the plane through \( P \) and the line \( (e_2x + e_3y + 1 = 0) \), the invariance of the denominators of transformation equations is not sufficient to assure invariance of the shape of the transformed spectrum locus. The chief subject of this paper is the investigation of the other conditions which, together with the invariance of the denominator, are necessary and sufficient to maintain the shape of the spectrum locus unchanged.

If two new variables, \( s \) and \( t \), are defined, such that
\[
\begin{align*}
s &= x/(e_2x + e_3y + 1), \quad (3) \\
t &= y/(e_2x + e_3y + 1), \quad (4)
\end{align*}
\]

then in terms of these variables the transformation Eqs. (1) and (2) become
\[
\begin{align*}
u &= e_2s + e_4t + e_6, \quad (5) \\
v &= e_2s + e_4t + e_6. \quad (6)
\end{align*}
\]

The coefficients, \( e_1, e_2, e_4, e_6 \), are related to the coefficients \( e_1 \cdots e_8 \) by the following expressions:
\[
\begin{align*}
e_1 &= c_1 - c_6c_7, \quad (7) \\
e_2 &= c_2 - c_6c_8, \quad (8) \\
e_4 &= c_4 - c_6c_7, \quad (9) \\
e_6 &= c_6 - c_6c_8. \quad (10)
\end{align*}
\]

The following equations* for the loci of constant \( s \) and constant \( t \) can be derived from Eqs. (5) and (6):
\[
\begin{align*}
u &= (e_3/e_1)u + (e_1 - e_4c_3/e_1)s + c_3 - c_6c_8/e_1, \quad (11) \\
v &= (e_3/e_1)u + (e_1 - e_4c_3/e_1)t + c_3 - c_6c_8/e_1. \quad (12)
\end{align*}
\]

From Eq. (11) it may be seen that the loci of constant values of \( s \) are parallel straight lines in the \((u, v)\)-plane, and the spacings of these loci are directly proportional to the differences of the corresponding values of \( s \). Similarly, from Eq. (12) it may be seen that the loci of constant values of \( t \) are parallel straight lines, the spacings of which are directly proportional to the differences of the values of \( t \). If invariance of the shape of the transformed spectrum locus is to be obtained, the ratio of the spacings of the loci of constant \( s \) to the spacings of the loci of constant

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* The remainder of this derivation, down to condition (14), was suggested by D. B. Judd as an abridged substitute for the derivation originally submitted to him by the author. The author is grateful to Dr. Judd for his painstaking care in checking this derivation, and for this and many other suggestions.
l must be maintained as must also the angle between the lines of constant $s$ and the lines of constant $t$.

The condition for invariance of this angle is of the familiar form: $(m_1 - m_2) / (1 + m_1 m_2)$ invariant, in which $m_1$ and $m_2$ are the slopes of the lines of constant $s$ and of constant $t$, respectively. Equations (11) and (12) show these slopes to be: $m_1 = e_b / e_y$, and $m_2 = e_y / e_x$. Therefore, the condition for the invariance of the angle between the lines of constant $s$ and constant $t$ becomes:

$$(e_y^2 - e_x^2) / (e_y^2 + e_x^2),$$

invariant. (13)

The condition for invariance in the ratio of the spacings of the loci of constant $s$ to the spacings of the loci of constant $t$ may be found by evaluating the perpendicular distance between the $s$ lines and the $t$ lines, respectively, for a unit change in $s$ or $t$. It is found that the distance between the line, $s = 0$, and the line, $t = 1$, is

$$(e_y^2 - e_x^2) / (e_y^2 + e_x^2),$$

and that the distance between the line, $s = 0$, and the line, $t = 1$, is

$$(e_y^2 - e_x^2) / (e_y^2 + e_x^2).$$

The condition for invariance in the ratio of these two spacings may therefore be written:

$$(e_y^2 + e_x^2) / (e_y^2 + e_x^2),$$

invariant. (14)

In summary, the shape of the spectrum locus is the same for all transformations for which $c_1$, $c_2$, and the expressions (13) and (14) are all invariant.

When the point, $x = u$, $y = v$ is transformed into the point $u = 0$ and $v = 0$, the constants $c_1$ and $c_2$ both vanish, and then

$$e_1 = c_1, \quad e_2 = c_2, \quad e_3 = c_3, \quad \text{and} \quad e_5 = c_5.$$

LIMITATIONS OF PROJECTIVE TRANSFORMATIONS

The limitations of projective transformations are not generally recognized. Those limitations are pertinent to the discussion of the possibility of representing the noticeability of color differences in terms of distance in some projective transformation of color-mixture diagrams. The following analysis will demonstrate that such a representation will be possible only if the increments of distance, $\Delta S$, along any straight line in the $(x, y)$ diagram corresponding to equally noticeable differences of colors are proportional to the square of the distance from some point on that line.

If the line in the $(x, y)$ diagram is inclined at the angle $\phi$ to the $y = 0$ axis, if $x_0$ is the $x$ coordinate of the intercept of that line with the $y = 0$ axis, and if $S$ is the distance along the line measured from that intercept, then

$$x = S \cos \phi + x_0,$$

$$y = S \sin \phi.$$

The values of $u$ and $v$ corresponding to values of $S$ along that straight line are given by

$$u = \left( (a_1 S + a_2) / (S + a_0) \right),$$

$$v = \left( (a_3 S + a_4) / (S + a_0) \right),$$

where

$$a_1 = (c_1 \cos \phi + c_2 \sin \phi) / (c_1 \cos \phi + c_3 \sin \phi),$$

$$a_2 = (c_1 c_3 x_0) / (c_1 \cos \phi + c_3 \sin \phi),$$

$$a_3 = (c_1 \cos \phi + c_3 \sin \phi) / (c_1 \cos \phi + c_3 \sin \phi),$$

$$a_4 = (c_1 c_3 x_0) / (c_1 \cos \phi + c_3 \sin \phi),$$

$$a_5 = (1 + c_3 x_0) / (c_1 \cos \phi + c_3 \sin \phi).$$

Equations (17) and (18) can be solved for $S$:

$$S = \left( (a_2 \alpha \omega \nu) / (\alpha \omega) - (a_1 \alpha \omega \nu) / (\nu \alpha \omega) \right).$$

If $T$ represents the distance along the corresponding line in the $(u, v)$ diagram, and if $\theta$ is the inclination of that line to the $v = 0$ axis, then $\Delta u = \Delta T \cos \theta$, and $\Delta v = \Delta T \sin \theta$. If the uniform spacing of the points representing equally noticeable different colors is accomplished by the use of the $(u, v)$ diagram, then $\Delta T$ will be constant for all pairs of equally noticeable different colors. Along any straight line, $\Delta S$ will also be constant for all such pairs of colors. The corresponding increments of $S$ will be given by

$$\Delta S = (dS/du) \Delta u = (dS/du) \Delta T \cos \theta.$$ (25)

From Eq. (24) the derivative can be found:

$$dS/du = (\alpha \omega \nu - \alpha \omega) / (\nu - \alpha \omega).$$ (26)
The value of $u$ given by Eq. (17) can be substituted in Eq. (26).

$$dS/du = (S+a_b)/(a_n u - a_b).$$  (27)*

When $\tan \phi = -c_1/c_6$, then the constants, $a_1$, $a_5$, $a_6$, become infinitely great, but the limit of $dS/du$, as $\tan \phi$ approaches $-c_1/c_6$, is finite, and is independent of $S$.

Limit, when $\tan \phi$ approaches $-c_1/c_6$

$$dS/du = (1 + c_5 c_6) (c_6^2 + c_5^2)/(c_6 c_5 - c_5 c_1).$$  (28)*

Along any straight line in the $(u, v)$ diagram, equal increments of $u$, given by Eq. (25), should correspond to equally noticeable differences of color. Therefore, the increments, $\Delta S = (dS/du)\Delta u$, of $S$ along the straight line in the $(x, y)$ diagram should correspond to equally noticeable color differences. If a color-mixture diagram having uniform chromaticity scales can be constructed, then increments of $\Delta S$ in the $(x, y)$ diagram proportional to

$$(S+a_b)^2 \cos \theta/(a_n a_5 - a_b)$$

should correspond to equally noticeable color differences. This means that if the desired representation of color differences is possible in a projective transformation of the $(x, y)$ diagram, then the curve representing $\Delta S$ as a function of $S$ along any straight line in the $(x, y)$ diagram must be a portion of a parabola with a vertical axis. The vertex (at which $\Delta S=0$) will occur at the value of $S$ corresponding to the intersection of the line with the line $(c_6 x + c_5 y + 1 = 0)$ which is projected to infinity by the successful transformation. For any series of colors differing along a straight line parallel to the line $(c_6 x + c_5 y + 1 = 0)$, that is, for $\tan \phi = -c_1/c_6$, $\Delta S$ should be independent of $S$ and should be equal to

$$\Delta S = (1 + c_6 c_5) (c_6^2 + c_5^2) \Delta T \cos \theta/(c_6 c_5 - c_5 c_1).$$  (29)*

Since observed values of $\Delta S$ cannot be zero, the observed values of $\Delta S$ along any straight line in the $(x, y)$ diagram must be either independent of $S$, that is, constant, or their dependence on $S$ must be represented by a portion of one branch of a vertical parabola, if the representation of the observed data by equal distances in some transformed diagram is to be possible. If, for any straight line in the $(x, y)$ diagram, the dependence of observed variations of $\Delta S$ on $S$ cannot be represented adequately by one branch of a parabola, then the results of the observations cannot be represented adequately by equal distances in any projective transformation of the $(x, y)$ diagram. Very few series of observations permitting the calculation of comparable values of $\Delta S$ for straight lines in the $(x, y)$ diagram have been published. Figures 5 and 6 of reference 3 and, less directly, Figs. 8-11, and 18 of reference 9, indicate that the observations cannot be represented adequately in the manner just prescried. The adequate representation of the results of observations of color differences by equal distances in some projective transformation of the $(x, y)$ diagram therefore seems improbable.

In some experimental arrangements it is more convenient to observe series of colors equally noticeably different from one or more standard colors. The results of such observations can be represented by the locus of points in the $(x, y)$ diagram corresponding to the colors equally noticeably different from the standard. Such results can be represented within the accuracy of the observations by ellipses9-12 in the $(x, y)$ diagram. If several ellipses represent the results of comparable observations using several standard colors, then these ellipses can be transformed into equal-sized circles by some projective transformation only if the following conditions are satisfied:

*In the $(x, y)$ diagram, the common tangents of


all pairs of observed ellipses must be either parallel or they must intersect on some one straight line. If the common tangents are parallel, they must also be parallel to the straight line on which the non-parallel pairs of common tangents intersect.

These conditions are consequences of the fact that the common tangents of equal-sized circles are parallel lines, and if such circles adequately represent the observations in the \((u, v)\) diagram, then the projective transformations of these common tangents onto the \(x, y\) plane must intersect on the line \((c_1 x + c_2 y + 1 = 0)\), which is projected to infinity in the \(u, v\) plane. In general, all lines of any parallel set in the \((u, v)\) diagram correspond to lines in the \((x, y)\) diagram which converge through a single point on the line \((c_1 x + c_2 y + 1 = 0)\).

If the conditions just stated are satisfied, then the straight-line locus of the intersection points of the common tangents of the ellipses in the \((x, y)\) diagram is the line \((c_1 x + c_2 y + 1 = 0)\). The left-hand member of the equation for this line is the denominator of the transformation equations for the desired uniform chromaticity scale diagram. If the conditions cited above are not fulfilled within a tolerance that corresponds to the uncertainty of the observations, then equally noticeable color differences cannot be represented adequately by equal distances in any projective transformation of the \((x, y)\) diagram.

Intrinsic Brightness as a Factor in Discomfort from Glare

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IT has been demonstrated in many ways that the loss in visual efficiency due to the presence of a glare-source within the visual field depends primarily upon the illumination at the eyes rather than upon the intrinsic brightness of the glare-source.\(^1\) Thus if a small but bright source produces the same illumination at the eyes as a large source of relatively low brightness, the reduction in visibility due to glare will be substantially the same in both cases. However, if the effect of glare is appraised from the viewpoint of ocular discomfort or ease of seeing, the foregoing generalization is not necessarily valid. In fact, it is obvious in extreme cases that the discomfort arising from glare sources within the visual field depends not only upon the illumination at the eyes but also upon the intrinsic brightness of the glare source. If the conditions of observation are so arranged that equal illuminations are produced at the eyes from sources differing in brightness and area, the smallest and consequently the brightest source will seem to be most objectionable. The purposes of the present investigation were (1) to test the validity of this introspective conclusion by means of the criterion of rate of involuntary blinking;\(^2\) and (2) to obtain some data with respect to the magnitudes of the complementary variables of brightness and area which are definitely associated with ocular discomfort or ease of seeing.

Previously we had applied the blink-rate criterion to actual lighting conditions in three offices, with encouraging results.\(^3\) The visual conditions involved in the present investigation are indicated diagrammatically in the accompanying illustration. The subjects were seated at a table placed in the center of a typical office illuminated by a totally indirect lighting system. The latter produced an illumination of 3.5 footcandles at the eyes and 10 footcandles upon the book to be read. The “glare source” consisted of an opal glass surface which could be uniformly illumina-

