Rigid Body Rotating

The total kinetic energy of a rigid body is given by the sum of the kinetic energy due to the translation of the center of mass and the kinetic energy from rotation calculated with respect to the center of mass (i.e., \( T = T_{\text{translation}} + T_{\text{rotation}} \)). The rotational kinetic energy is:

\[
T_{\text{rotation}} = \frac{1}{2} \sum_{i} m_i (\vec{\omega} \times \vec{r}_i)^2 = \frac{1}{2} \sum_{\alpha, \beta} \omega_{\alpha} \omega_{\beta} I_{\alpha\beta}, \quad I_{\alpha\beta} \equiv \sum_{i} m_i (r_i^2 \delta_{\alpha\beta} - r_{i,\alpha} r_{i,\beta})
\]

where \( I_{\alpha\beta} \) are the components of the moment of inertia tensor. The moment of inertia tensor has units of mass x length^2 and is a second rank tensor. A second-rank tensor is a matrix that is invariant under space rotation after dotting with two vectors.

The principal axis system is a body coordinate system which diagonalize the moment of inertial tensor. The 3 diagonal elements are the principal moments of inertia. For a sufficiently symmetric body the direction of the principal axes are obvious.

We can change the coordinates of the moment of inertia tensor by doing the normal orthogonal transformation so that:

\[
\mathbf{I}' = \mathbf{U} \mathbf{I} \mathbf{U}^T, \quad \text{where} \quad \omega' = \mathbf{U} \omega \quad \text{and} \quad \omega = \mathbf{U}^T \omega'
\]

Displaced Axis Theorem

Given that we know the moment of inertia tensor rotating about its center of mass \( (I_{\text{cm}}) \), the moment of inertia tensor about a point displaced by \( \vec{a} \) is:

\[
I_{\vec{a}} = I_{\text{cm}} + M (\vec{a}^2 \delta_{\alpha\beta} - a_{\alpha} a_{\beta})
\]

If the displacement \( \vec{a} \) is along one of the principal axes, then this equation reduces to the more familiar parallel axis theorem

\[
I_{\vec{a}} = I_{\text{cm}} + Ma^2
\]

Angular Momentum

The angular momentum due to the rotation of a body is given by

\[
\mathbf{L}_{\text{rotation}} = \sum_{i} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \mathbf{I} \cdot \omega
\]

Euler Equations

Using the relation between time derivatives between space and body frame, we have the Euler Equations

\[
\begin{align*}
I_1 \frac{d\omega_1}{dt} &= \omega_2 \omega_3 (I_2 - I_3) \\
I_2 \frac{d\omega_2}{dt} &= \omega_3 \omega_1 (I_3 - I_1) \\
I_3 \frac{d\omega_3}{dt} &= \omega_1 \omega_2 (I_1 - I_2)
\end{align*}
\]
**Euler Angles**

Euler Angles are a systematic set of 3 rotations performed one after another to describe any rotation in 3D space: first rotate about the \( z' \) axis by an angle \( \phi \), then about \( \xi' \) by an angle \( \theta \), then finally about \( \zeta' \) by an angle \( \psi \).

Diagram from Professor Michael Cross Lecture slides

**Problem 1:** HF 8-6 (\( I \) for a circular cylinder) Find the inertia tensor, principal axes, and principal moments for a circular cylinder of radius \( R \) and height \( h \).

**Problem 2:** A torsion pendulum consists of a vertical wire attached to a mass that may rotate about the vertical. Consider three torsion pendulums that consist of identical wires from which identical homogeneous solid cubes are hung. One cube is hung from a corner, one from midway along an edge, and one from the middle of a face, as shown in the figure. What are the ratios of the periods of the three pendula?

**Problem 3:** HF 8-15 (Rotating a rectangular plate) Using Euler Equations \( \tau_1 = I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3 \), we will solve for the torque needed to rotate a rectangular plate about a diagonal with a constant angular velocity \( \vec{\omega} \).

a) What is \( \vec{\omega} \) in the principal axis frame?

b) Why is the (body) coordinate system chosen above useful? Why not have it tilted at some angle in the XY plane?

c) Using the fact that \( \vec{\omega} \) is constant in the body frame, use the Euler Equations to solve for the torque components. Prove that

\[
\vec{\tau} = \frac{Ma\omega^2(b^2 - a^2)}{12(a^2 + b^2)} \hat{k}
\]

(You can assume the plate has zero thickness.)

d) Note that \( \vec{\tau} = 0 \) if \( a = b \). Explain. If \( b > a \) draw a vector diagram showing \( \vec{I} \) and \( \vec{\omega} \) in the body system. Then explain why you need to exert a torque about the plate diagonal axis in order to rotate the plate.