Hamiltonian Mechanics
From last week, we have the Euler-Lagrange equation.

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \]

This will be useful for solving a system where we have holonomic constraints and conservative forces.

From the Lagrangian, we can define the Hamiltonian to be

\[ H = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L. \]

H is often written as T + V, the sum of kinetic and potential energy. However, it is important to remember that this is only true in the case of a system having only scleronomic constraints (constraints that do not have explicit time dependence).

If a coordinate (ie, x, y,...) does not show up in the Lagrangian but its times derivative does, we call that coordinate ignorable, or cyclic. Ignorable or cyclic coordinates in a system will result in some conservation law of its corresponding momentum.

In an optimizing problem, given that we want to find an extremum of the functional \( I = \int_{x_0}^{x_1} F \left( y, \frac{dy}{dx}, x \right) dx \), (a functional is a function of functions, ie. in our case, \( F \) is a function of \( y \) and \( y \) is a function of \( x \)), the general solution is just

\[ \frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{\partial F}{\partial \frac{dy}{dx}} \right) \]

It turns out that if we replace our functional with the Lagrangian and integrate it with respect to time, the solution that optimizes the integral \( S[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t) dt \), (called action), is just the Euler-Lagrange equation. Since we know that the Euler-Lagrange equation is equivalent to Newton’s Laws, we can find the equation of motion of a system by optimizing the action integral.

In a system with more than one degree of freedom, \( N \), that are all independent from one another, by setting changes in the action integral to 0, we can obtain \( N \) Euler-Lagrange equations to obtain the \( N \) equations of motion of the system.

**Problem 1**: HF 1-22 (Box sliding horizontally) A box of mass M slides horizontally on a frictionless surface. The distance of the box’s center of mass from the origin is denoted by \( X \). Suspended from inside the center of the box is a pendulum of length \( l \) at the bottom of which is a mass m. All the motion takes place in the XY plane. What is the Lagrangian for this system? What are the EOMs?

**Problem 2**: Consider the function \( y(x) = x + \alpha \sin x \). Find \( \alpha \) to minimize the distance between the limit \( x = 0 \) and \( x = 2\pi \).

**Problem 3**: Brachistochrone: Consider a particle moving in a constant force field starting at rest from some point \((x_1, y_1)\) to some lower point \((x_2, y_2)\). Find the path that allows the particle to accomplish the transit in the least possible time.