1.) (5 point) Suppose that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent. Use the fact that $X \sim \mathcal{N}(\mu, \sigma^2)$ has moment generating function $M_X(t) = e^{\mu t + \frac{1}{2} \sigma^2 t^2}$ to derive the distribution of $X_1 + X_2$.

Solution. The moment generating function of $X_1 + X_2$ is

$$M_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}] = E[e^{tX_1} e^{tX_2}] = E[e^{tX_1}] E[e^{tX_2}]$$

(here we used independence) $= M_{X_1}(t) M_{X_2}(t) = e^{\mu_1 t + \frac{1}{2} \sigma_1^2 t^2} e^{\mu_2 t + \frac{1}{2} \sigma_2^2 t^2} = e^{(\mu_1 + \mu_2) t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2) t^2},$

which is the MGF of a $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ random variable. Since the distribution of a random variable is completely characterized by its moment generating function, it follows that $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

2.) (5 point) A simple random sample of 400 people was taken from a city of 100,000 people. There were 320 males and 80 females in the sample. Estimate the population proportion $p$ of males in the city, and use the result to approximate the standard deviation of your estimator.

Solution. Let $X_1, \ldots, X_n$ be Bernoulli random variables, where $n = 400$. Since it is a simple random sample from the population and because the population size is much larger than the sample size, $X$-s may be considered independent. Assume they also have a common distribution which is the true population distribution, and that $n$ is large enough. Let $\hat{p}$ be the proportion estimator:

$$\hat{p} = \frac{X_1 + \cdots + X_n}{n}.$$

Then $E[\hat{p}] = p$ and

$$\text{Var}[\hat{p}] = \frac{1}{n^2} \text{Var}(X_1 + \cdots + X_n) = \frac{p(1-p)}{n}.$$

We observe $\hat{p} = 320/400 = 0.8$, hence our estimate of $p$ is 0.8 and the standard deviation of our estimator can be approximated by $\sqrt{0.8 \cdot 0.2/400} = 0.02$. 