This was a 20 minute quiz worth 10 points.

The Weak Law of Large Numbers says, that if $X_1, X_2, \cdots$ is a sequence of i.i.d. random variables with $E(|X_1|) < \infty$, then $\sum_{i=1}^{n} X_i/n$ converges in probability to $\mu = E(X_1)$.

Use Chebyshev’s inequality to show that if $X_1, X_2, \cdots$ is a sequence of uncorrelated random variables, all having mean $E(X_i) = \mu$ and variance $\text{Var}(X_i) = \sigma^2 < \infty$, then $\sum_{i=1}^{n} X_i/n$ converges in probability to $\mu$.

**Solution.** Let 

$$Y_n = \frac{X_1 + \cdots + X_n}{n}.$$ 

Then 

$$EY_n = E\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} EX_i = \frac{1}{n} n EX_1 = \mu,$$

and since $X_1, X_2, \cdots$ are uncorrelated, $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$, hence 

$$\text{Var}Y_n = E\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)^2 = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}X_i = \frac{1}{n^2} n \text{Var}X_1 = \frac{\sigma^2}{n}.$$

According to Chebyshev’s inequality, for any $\varepsilon > 0$

$$P(|Y_n - EY_n| \geq \varepsilon) \leq \frac{\text{Var}Y_n}{\varepsilon^2} \iff P\left(\left|\frac{1}{n} \sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n \varepsilon^2}.$$ 

Since probabilities are always nonnegative, taking the limit as $n \to \infty$, one gets 

$$0 \leq \lim_{n \to \infty} P\left(\left|\frac{1}{n} \sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n \varepsilon^2} = 0.$$ 

It follows that

$$\lim_{n \to \infty} P\left(\left|\frac{1}{n} \sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right) = 0.$$