This was a 20 minute quiz worth 10 points.

Let $X$ be an exponentially distributed random variable with mean 1, i.e.

$$F_X(x) = \begin{cases} 1 - e^{-x}, & x \geq 0, \\ 0, & x < 0 \end{cases}, \quad f_X(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & x < 0 \end{cases},$$

where $F_X(x)$ is the c.d.f. of $X$ and $f_X(x)$ is its p.d.f. Let $Y = e^{aX}$.

1.) (1 point) What is the support (the set of possible values) of $Y$?

Solution. Since $X$ can have values in $[0, \infty)$, $Y = e^{aX}$ can have possible values in $[1, \infty)$.

2.) (3 points) What is the p.d.f. of $Y$?

Solution. First, let’s find the c.d.f. of $Y$:

$$F_Y(y) = P(Y \leq y) = P(e^{aX} \leq y) = P(X \leq (\log y)/a) = 1 - e^{-(\log y)/a} = 1 - \frac{1}{y^{1/a}}, \quad y \geq 1.$$

Then the p.d.f. of $Y$ is

$$f_Y(y) = F_Y'(y) = \frac{1}{a y^{1/a+1}}, \quad y \geq 1.$$

3.) (3 points) What is the expected value of $Y$?

Solution.

$$E[Y] = \int_1^{\infty} y f_Y(y) \, dy = \int_1^{\infty} \frac{1}{a y^{1/a}} \, dy = \begin{cases} \infty, & \text{if } a \geq 1, \\ \frac{a}{a-1}, & \text{if } a < 1. \end{cases}$$

4.) (3 points) What is the joint c.d.f. of $X$ and $Y$?

Solution. For $x \geq 0$ and $y \geq 1$,

$$P(X \leq x, \ Y \leq y) = P(X \leq x, \ e^{aX} \leq y) = P(X \leq x, \ X \leq (\log y)/a) = P(X \leq \min(x, (\log y)/a)) = 1 - e^{-\min(x, (\log y)/a)} = 1 - \max \left( e^{-x}, \frac{1}{y^{1/a}} \right).$$