Lecture 1. ABC of Probability
Agenda

- Sample Spaces
- Realizations, Events
- Axioms of Probability
- Probability on Finite Sample Spaces
  - Example: B-day Problem
- Independent Events
- Summary
Sample Spaces, Realizations, Events

**Probability Theory** is the mathematical language for uncertainty quantification. The starting point in developing the probability theory is to specify sample space $\Omega$ = the set of possible outcomes.

**Definition**
- The **sample space** $\Omega$ is the set of possible outcomes of an “experiment”
- Points $\omega \in \Omega$ are called **realizations**
- **Events** are subsets of $\Omega$

Next, to every event $A \subset \Omega$, we want to assign a real number $P(A)$, called the probability of $A$. We call function $P : \{\text{subsets of } \Omega\} \to \mathbb{R}$ a probability distribution.

We don’t want $P$ to be arbitrary, we want it to satisfy some natural properties (called **axioms of probability**):
1. $0 \leq P(A) \leq 1$ (Events range from never happening to always happening)
2. $P(\Omega) = 1$ (Something must happen)
3. $P(\emptyset) = 0$ (Nothing never happens)
4. $P(A) + P(\overline{A}) = 1$ ($A$ must either happen or not-happen)
5. $P(A + B) = P(A) + P(B) - P(AB)$
Probability on Finite Sample Spaces

Suppose that the **sample space is finite** $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$.

**Example:**
If we toss a die twice, then $\Omega$ has $n = 36$ elements:

$$\Omega = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$$

If each outcome is **equally likely**, then $\mathbb{P}(A) = |A|/36$, where $|A|$ denotes the number of elements in $A$.

**Test question:** What is the probability that the sum of the dice is 11?

**Answer:** $2/36$, since the are two outcomes that correspond to this event: $(5, 6)$ and $(6, 5)$.

In general, if $\Omega$ is **finite** and if each outcome is equally likely, then

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

To compute the probability $\mathbb{P}(A)$, we need to **count** the number of points in an event $A$. Methods for counting points are called **combinatorial methods**.
Example: Birthday Problem

Suppose that a room of people contains \( n \) people. What is the probability that at least two of them have a common birthday?

Assume that

- Every day of the year is equally likely to be a birthday
- There are 365 days in the year (disregard leap years)

Then

\[
\begin{align*}
\Omega &= \{ \omega = (x_1, \ldots, x_n) : x_i = 1, 2, \ldots, 365 \}, \quad |\Omega| = 365^n \\
A &= \{ \omega \in \Omega : x_i = x_j \text{ for some } i \neq j \} \\
\bar{A} &= \{ \omega \in \Omega : x_i \neq x_j \text{ for all } i, j \}, \quad |\bar{A}| = 365 \times 364 \times \ldots \times (365 - n + 1)
\end{align*}
\]

\[
\mathbb{P}(A) = 1 - \frac{365 \times 364 \times \ldots \times (365 - n + 1)}{365^n}
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \mathbb{P}(A) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.016</td>
</tr>
<tr>
<td>23</td>
<td>0.507</td>
</tr>
<tr>
<td>32</td>
<td>0.753</td>
</tr>
<tr>
<td>42</td>
<td>0.91</td>
</tr>
<tr>
<td>56</td>
<td>0.988</td>
</tr>
</tbody>
</table>
Example: Birthday Problem

![Graph showing the probability of having at least two people with a common birthday as a function of the number of people in a room. The probability increases as the number of people increases.]

Konstantin Zuev (USC)  
Math 408, Lecture 1  
January 16, 2013  
6 / 9
Independent Events

If we flip a fair coin twice, then the probability of two heads is $\frac{1}{2} \times \frac{1}{2}$. We multiply the probabilities because we regard the two tosses as independent. We can formalize this useful notion of independence as follows:

**Definition**

Two events $A$ and $B$ are **independent** if

$$P(AB) = P(A)P(B)$$

Independence can arise in two distinct ways:

1. We explicitly assume that two events are independent. For example, in tossing a coin twice, we usually assume that the tosses are independent which reflects the fact that the coin has no memory of the first toss.

2. We derive independence of $A$ and $B$ by verifying that $P(AB) = P(A)P(B)$. For example, in tossing a fair die, let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$. Are $A$ and $B$ independent?

Yes! Since $P(A) = 1/2$, $P(B) = 2/3$, $AB = \{2, 4\}$, $P(AB) = 1/3 = (1/2) \times (2/3)$
Examples

- Suppose that $A$ and $B$ are disjoint events, each with positive probability. Can they be independent?
  Answer: No! $\mathbb{P}(AB) = \mathbb{P}(\emptyset) = 0$, but $\mathbb{P}(A)\mathbb{P}(B) > 0$

- Two people take turns trying to sink a basketball into a net.
  - Person 1 succeeds with probability $1/3$
  - Person 2 succeeds with probability $1/4$

  What is the probability that person 1 succeeds before person 2?
  Answer: $2/3$
Summary

- **The sample space** $\Omega$ is the set of possible outcomes of an “experiment”
- Points $\omega \in \Omega$ are called **realizations**
- **Events** are subsets of $\Omega$
- **Properties (axioms) of probability:**
  - $0 \leq \mathbb{P}(A) \leq 1$ (Events range from never happening to always happening)
  - $\mathbb{P}(\Omega) = 1$ (Something must happen)
  - $\mathbb{P}(\emptyset) = 0$ (Nothing never happens)
  - $\mathbb{P}(A) + \mathbb{P}(\overline{A}) = 1$ (A must either happen or not-happen)
  - $\mathbb{P}(A + B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$
- $A$ and $B$ are **independent** if $\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$
- Independence is sometimes **assumed** and sometimes **derived**.
- **Disjoint** events with **positive probability** are **not independent**.