**Problem 1.** Determine the longest interval in which the given initial value problem is certain to have a unique solution:

$$(\cos^2 x - 1)y'' + y' \ln|x-4| + \sin x = 1, \qquad y(5) = 0, \quad y'(5) = 1.$$

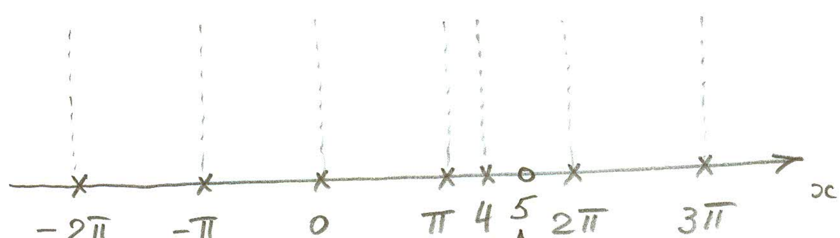
First, let us write the equation in the standard form:

$$y'' + y' \cdot \frac{\ln|x-4|}{\cos^2 x - 1} + \frac{\sin x}{\cos^2 x - 1} = \frac{1}{\cos^2 x - 1}$$

Points of discontinuity of the coefficients

$$\frac{\ln|x-4|}{\cos^2 x - 1}, \quad \frac{\sin x}{\cos^2 x - 1}, \quad \text{and} \quad \frac{1}{\cos^2 x - 1}$$

are : $x \neq 4$ and $\cos x \neq \pm 1 \iff x \neq 0, \pm\pi, \pm 2\pi, \ldots$




initial conditions $\begin{cases} y(5) = 0 \\ y'(5) = 1 \end{cases}$

$\Rightarrow$ the solution is certain to exist

on $(4, 2\pi)$

$$\boxed{(4, 2\pi)}$$

**Problem 2.** Solve the given initial value problem

$$y'' - 2y' + 10y = 0, \quad y(0) = 1, \quad y'(0) = -2.$$

The characteristic equation is $\lambda^2 - 2\lambda + 10 = 0$

$$\Rightarrow \quad \lambda_{1,2} = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$$

The general solution is then

$$y(t) = C_1 e^t \cos 3t + C_2 e^t \sin 3t$$

Let us find $C_1$ and $C_2$ from the initial conditions

$$y(0) = 1 \quad \Rightarrow \quad C_1 = 1$$

$$y'(t) = C_1 e^t \cos 3t - 3C_1 e^t \sin 3t + C_2 e^t \sin 3t + 3C_2 e^t \cos 3t$$

$$y'(0) = -2 \quad \Rightarrow \quad C_1 + 3C_2 = -2 \quad \Rightarrow \quad C_2 = -1$$

Thus, the solution of the IVP is

$$y(t) = e^t \cos 3t - e^t \sin 3t$$

$$\boxed{e^t \cos 3t - e^t \sin 3t}$$

**Problem 3.** Find a particular solution of the following 2nd order ODE using the method of undetermined coefficients:

$$y'' + 2y' + 2y = 10 + e^{-t}\cos t$$

1) $y'' + 2y' + 2y = 10$ A particular solution of this equation is $\boxed{Y_1(t) = 5}$

2) $y'' + 2y' + 2y = e^{-t}\cos t$

Let us look for a solution in the form $Y_2(t) = t^s\left[Ae^{-t}\cos t + Be^{-t}\sin t\right]$

where $s = \#$ of times $\lambda = -1+i$ is a root of the characteristic eq.

Thus, $s = 1$, and

$\lambda^2 + 2\lambda + 2 = 0$

$\lambda_{1,2} = \dfrac{-2 \pm \sqrt{4-8}}{2} = \underline{-1 \pm i}$

- $Y_2(t) = Ate^{-t}\cos t + Bte^{-t}\sin t$
- $Y_2'(t) = Ae^{-t}\cos t - Ate^{-t}\cos t - Ate^{-t}\sin t + Be^{-t}\sin t - Bte^{-t}\sin t + Bte^{-t}\cos t$
  $= Ae^{-t}\cos t + Be^{-t}\sin t + (B-A)te^{-t}\cos t - (A+B)te^{-t}\sin t$
- $Y_2''(t) = -Ae^{-t}\cos t - Ae^{-t}\sin t - Be^{-t}\sin t + Be^{-t}\cos t + (B-A)e^{-t}\cos t - (B-A)te^{-t}\cos t$
  $- (B-A)te^{-t}\sin t - (A+B)e^{-t}\sin t + (A+B)te^{-t}\sin t - (A+B)te^{-t}\cos t =$
  $= 2(B-A)e^{-t}\cos t - 2(A+B)e^{-t}\sin t - 2Bte^{-t}\cos t + 2Ate^{-t}\sin t$

So ☺

$2(B-A)e^{-t}\cos t - 2(A+B)e^{-t}\sin t - 2Bte^{-t}\cos t + 2Ate^{-t}\sin t$

$+ 2Ae^{-t}\cos t + 2Be^{-t}\sin t + 2(B-A)te^{-t}\cos t - 2(A+B)te^{-t}\sin t$

$+ 2Ate^{-t}\cos t + 2Bte^{-t}\sin t = e^{-t}\cos t$

$2Be^{-t}\cos t - 2Ae^{-t}\sin t = e^{-t}\cos t$

$$\begin{cases} A = 0 \\ B = \frac{1}{2} \end{cases}$$

$Y_2(t) = \frac{1}{2}te^{-t}\sin t$

$Y = Y_1 + Y_2 = 5 + \frac{1}{2}te^{-t}\sin t$

$$\boxed{Y(t) = 5 + \frac{1}{2}te^{-t}\sin t}$$