Math 245 - Mathematics of Physics and Engineering I

Lecture 40. Periodic Solutions and Limiting Cycles

April 25, 2012

Periodic Solutions

In this Lecture, we discuss the possible existence of periodic solutions of two-dimensional autonomous systems

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \tag{1}$$

- A periodic solution $\mathbf{x}(t)$ is a solution that satisfies the relation

$$\mathbf{x}(t+T) = \mathbf{x}(t) \tag{2}$$

  for some constant $T > 0$ that is called the period.
- The trajectories of periodic solutions are closed curves in the phase plane.
- Periodic solutions often play an important role in physical problems because they represent phenomena that occur repeatedly.
- Critical point is a special case of a periodic solution.
- Consider a linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$
  - If $\lambda_{1,2} = \pm i\beta$, then all solutions are periodic.
  - Otherwise, there are no periodic solutions (except for $\mathbf{x} = \mathbf{0}$)

Example: Basic Analysis

Consider the following nonlinear system:

$$x' = x + y - x(x^2 + y^2), \qquad y' = -x + y - y(x^2 + y^2) \tag{3}$$

1. $(0,0)$ is the only critical point of (3)
2. The corresponding linear system (linearization of (3) at (0,0)) is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

3. $\det A = 2, \quad \mathrm{tr} A = 2, \quad D = (\mathrm{tr} A)^2 - 4 \det A = -4$
   $\Rightarrow$ $(0,0)$ is unstable spiral source for both the nonlinear system (3) and its linearization (4). Thus, any solution of (3) that starts near the origin will spiral away from the origin.

   Question: Will they spiral away to infinity?

Example: Introducing Polar Coordinates

Let us introduce polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta \tag{5}$$

Then, if $r \neq 0$, the original system is equivalent to

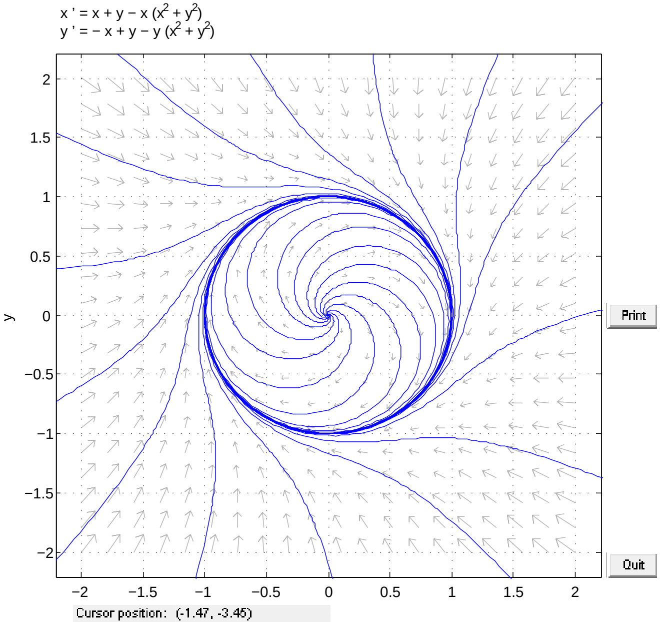
$$r' = r(1 - r^2), \qquad \theta' = -1 \tag{6}$$

- One solution of (6) is $r = 1,\ \theta = -t + \theta_0$. This is a periodic solution: a point moves clockwise around the unit circle.
- If $r \neq 0$, then the solution of (6) that satisfies the initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$ is given by

$$r(t) = \frac{1}{\sqrt{1 + (1/r_0^2 - 1)e^{-2t}}}, \qquad \theta(t) = -t + \theta_0 \tag{7}$$

  - if $r_0 < 1$, then $r \to 1$ from the inside as $t \to \infty$
  - if $r_0 > 1$, then $r \to 1$ from the outside as $t \to \infty$

Example: Phase Portrait



Remarks and Terminology

Remarks:

- It the example, the circle $r = 1$ not only corresponds to periodic solution, but also attracts other trajectories that spiral toward it as $t \to \infty$.
- A closed trajectory that attracts other trajectories is called a **limit cycle**. Thus, the circle $r = 1$ is a limit cycle.
- Orbital Stability
  - If all trajectories that start near a closed trajectory spiral toward the closed trajectory as $t \to \infty$, then the limit cycle is asymptotically stable.
  - If the trajectories on one side spiral toward the closed trajectory, while those on the other side spiral away as $t \to \infty$, then the limit cycle is semistable.
  - If the trajectories on both sides of the closed trajectory spiral away as $t \to \infty$, then the closed trajectory is unstable.

General Theorems

It is worthwhile to know general theorems concerning the existence or nonexistence of periodic solutions (i.e. closed trajectories) of nonlinear autonomous systems.

$$x' = F(x, y), \quad y' = G(x, y) \tag{8}$$

Theorem

Let F and G have continuous first partial derivatives in a domain $D \subset \mathbb{R}^2$. A closed trajectory of (8) must enclose at least one critical point which is not a saddle point.

Remark: This theorem is useful in a negative sense:

- If a given domain contains no critical points (or only saddle points), then there can be no closed trajectory lying entirely in this domain.

General Theorems

$$x' = F(x,y), \quad y' = G(x,y) \tag{9}$$

Theorem

Let F and G have continuous first partial derivatives in a simply connected domain $D \subset \mathbb{R}^2$ ("no holes"). If $F_x + G_y$ has the same sign throughout D, then there is no closed trajectory of (9) lying entirely in D.

Remark: If $F_x + G_y$ changes its sign, then no conclusion can be drawn.

Example:

$$x' = x + y - x(x^2 + y^2), \qquad y' = -x + y - y(x^2 + y^2)$$

Theorems says that there is no closed trajectory in the domain $\{r < 1/\sqrt{2}\}$.

Summary

- A periodic solution $\mathbf{x}(t)$ is a solution that satisfies the relation $\mathbf{x}(t+T) = \mathbf{x}(t)$ for some constant $T > 0$ that is called the period.
- The trajectories of periodic solutions are closed curves in the phase plane.
- For a linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$
  - If $\lambda_{1,2} = \pm i\beta$, then all solutions are periodic.
  - Otherwise, there are no periodic solutions (except for $\mathbf{x} = \mathbf{0}$)
- A closed trajectory that attracts other trajectories is called a **limit cycle**.

General Theorems

$$x' = F(x,y), \quad y' = G(x,y)$$

- Let F and G have continuous first partial derivatives in a domain $D \subset \mathbb{R}^2$.
  - A closed trajectory must enclose at least one critical point.
  - If it encloses only one critical point, the critical point cannot be a saddle point.
  - If D is simply connected and $F_x + G_x$ has the same sign throughout D, then there is no closed trajectory lying entirely in D.

Homework

<u>Homework:</u>

- Section 7.5
  - 3, 5, 11