Math 245 - Mathematics of Physics and Engineering I

Lecture 39. Competing Species

April 23, 2012

Competing Species

Suppose that in some closed environment there are two similar species competing for a limited food supply- for example, two species of fish in a pond that do not prey on each other but do compete for the available food.
Let $x(t)$ and $y(t)$ be the populations of the two species at time t. Assume that each population, in the absence of the other, is governed by a logistic equation (see Lecture 6):

$$\begin{aligned} x' &= x(\varepsilon_1 - \sigma_1 x) \\ y' &= y(\varepsilon_2 - \sigma_2 y) \end{aligned} \tag{1}$$

- ε_1 and ε_2 are the growth rates of the two populations
- ε_1/σ_1 and ε_2/σ_2 are their saturation levels

When both species are present, each will impinge on the available food supply for the other. In effect, they reduce each other's growth rate and suturation levels.

$$\begin{aligned} (\varepsilon_1 - \sigma_1 x) &\quad \rightsquigarrow \quad (\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ (\varepsilon_2 - \sigma_2 y) &\quad \rightsquigarrow \quad (\varepsilon_2 - \sigma_2 x - \alpha_2 x) \end{aligned} \tag{2}$$

- α_1 and α_2 are measures of the degree of interference between populations.

Competing Species

Thus we have the following system of equations:

$$\begin{aligned} x' =& x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ y' =& y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) \end{aligned} \qquad (3)$$

The values of the positive parameters $\varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2, \alpha_1$ and α_2 depend on the particular species sunder consideration and, in general, must be determined from observations.

Remarks:

- Although this model is extremely simple compared to the very complex relationships that exist in nature, it is still possible to acquire some insight into ecological principles from a study of this idealized problem.
- Similar models have also been used to study other competitive situations, for example, businesses competing in the same economic markets

Example

Consider the following system:

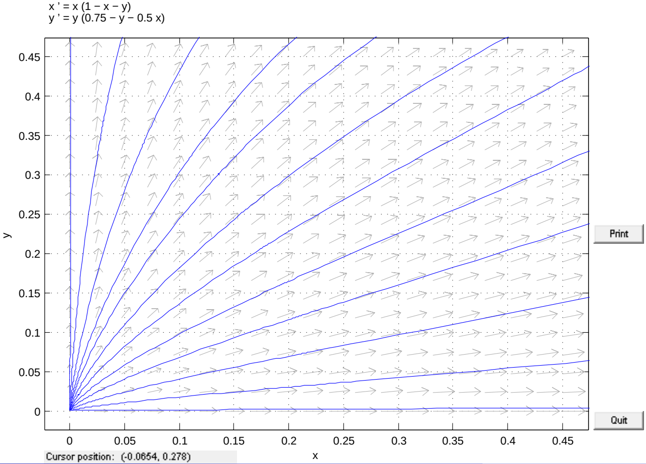
$$x' = x(1 - x - y)$$
$$y' = y(0.75 - y - 0.5x)$$

- Find all critical points.
  Answer: $(0,0), (1,0), (0,0.75)$, and $(0.5, 0.5)$
- Determine the type and stability of each critical point.

(0,0)

The corresponding linear system: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

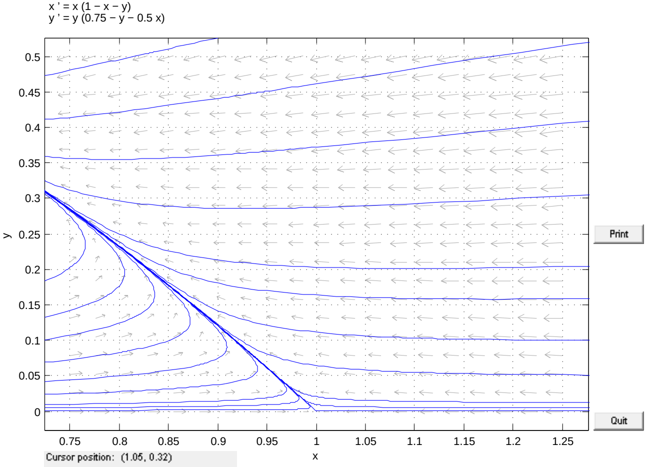
- $\det A = 0.75, \quad \mathrm{tr} A = 1.75, \quad D = (\mathrm{tr} A)^2 - 4 \det A = 0.0625$
- $\Rightarrow (0,0)$ is unstable nodal source for both nonlinear system and its linearization.



(1,0)

The corresponding linear system: $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

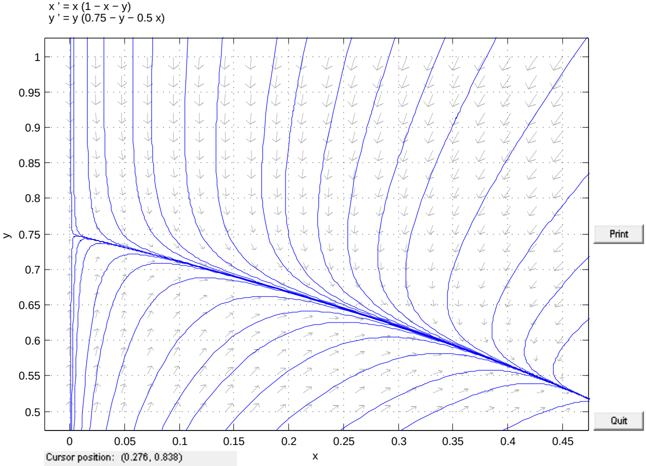
- $\det A = -0.25$
- $\Rightarrow (1,0)$ is unstable saddle point for both nonlinear system and its linearization.



(0,0.75)

The corresponding linear system: $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

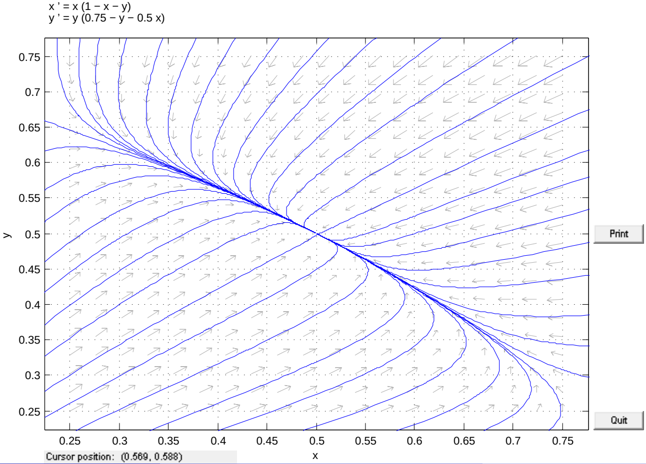
- $\det A = -0.1875$
- $\Rightarrow (1, 0)$ is unstable saddle point for both nonlinear system and its linearization.



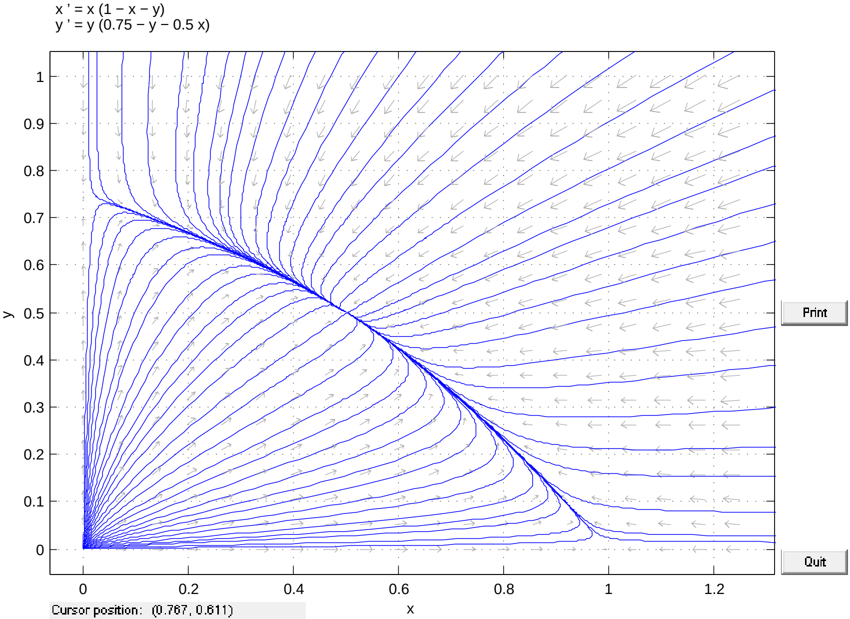
(0.5,0.5)

The corresponding linear system: $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

- $\det A = 0.1250, \quad \mathrm{tr}A = -1, \quad D = (\mathrm{tr}A)^2 - 4\det A = 0.5$
- $\Rightarrow (1,0)$ is asymptotically stable nodal sink for both nonlinear system and its linearization.



Phase portrait



Homework

Homework:

- Section 7.3
  - 1, 3, 5