Mathematics of Physics and Engineering I

Lecture 38. Almost Linear Systems

April 20, 2012
Agenda

- Stability Properties of Linear Systems
- Effect of Small Perturbations
- Linear Approximations to Nonlinear Systems
- Stability of Phase Portraits
- Summary and Homework
Linear Systems

In Lecture 14, we studied stability properties of linear systems

$$x' = Ax$$

If we assume that $\det A \neq 0$, then $x = 0$ is the only critical point of (1).
Linear Systems

Theorem

The critical point \( \mathbf{x} = 0 \) of the linear system \( \mathbf{x}' = \mathbf{A}\mathbf{x} \) is

- **stable**, but not asymptotically stable, if \( \lambda_1 \) and \( \lambda_2 \) are purely imaginary
- **asymptotically stable**, if \( \lambda_1 \) and \( \lambda_2 \) are
  - real and negative or
  - complex with negative real part
- **unstable**, if \( \lambda_1 \) and \( \lambda_2 \) are
  - real and at least one of them is positive or
  - complex with positive real part

Thus, both the type and stability properties are determined by the eigenvalues of the coefficient matrix \( \mathbf{A} \). In turn, the values of \( \lambda_1 \) and \( \lambda_2 \) depend on the coefficients, i.e. entities of \( \mathbf{A} \). In applications, these coefficients usually are obtained from measurements of certain physical quantities. Such measurements are often subject to small errors. Therefore, it is interesting to investigate the following question:

**can small perturbations of coefficients affect the stability of a critical point?**
Small Perturbations

The eigenvalues $\lambda_1$ and $\lambda_2$ are the roots of the polynomial equation

$$\det(A - \lambda I) = 0$$

(1)

It is possible to show that small perturbations of coefficients are reflected in small perturbations of the eigenvalues. There are only two sensitive situations:

1. **Most sensitive case:** $\lambda_{1,2} = \pm i\beta$ (the critical point is a stable center)
   $$\lambda_1 = i\beta \quad \leadsto \quad \hat{\lambda}_1 = \hat{\alpha} + i\hat{\beta}$$
   $$\lambda_2 = -i\beta \quad \leadsto \quad \hat{\lambda}_1 = \hat{\alpha} - i\hat{\beta}$$
   ▶ If $\hat{\alpha} = 0$ (very unlikely), then the critical point remains a stable center
   ▶ If $\hat{\alpha} > 0$, then the critical point becomes an unstable spiral source
   ▶ If $\hat{\alpha} < 0$, then the critical point becomes asymptotically stable spiral sink

2. **Less sensitive case:** $\lambda_1 = \lambda_2 = \lambda$, the critical point is a degenerate nodal sink ($\lambda < 0$) or a degenerate nodal source ($\lambda > 0$).
   $$\lambda_1 = \lambda_2 \quad \leadsto \quad \hat{\lambda}_1 \neq \hat{\lambda}_2$$
   ▶ If $\hat{\lambda}_{1,2}$ are real, then the critical node becomes a nodal sink/source
   ▶ If $\hat{\lambda}_{1,2}$ are complex, then the critical node becomes a spiral sink/source

In Case 1, both **stability** and **trajectories** are affected.

In Case 2, only **trajectories** are affected.
Linear Approximations to Nonlinear Systems

Our next goal: to investigate the behavior of trajectories of a nonlinear system $x' = f(x)$ near a critical point

Strategy: approximate the nonlinear system by a linear system (trajectories of linear systems are easy to describe)

Question: What does it mean for a nonlinear system to be approximately linear?

Consider the following system:

$$x' = Ax + g(x) \quad (4)$$

and suppose that $x = 0$ is an isolated critical point of (4). For system (4) to be close to the linear system $x' = Ax$, we must assume that $g(x)$ is small.

More precisely,

**Definition**

System (4) is called an **almost linear system** in the neighborhood of $x = 0$ if

- $g(x)$ has continuous partial derivatives
- As $x \rightarrow 0$,

$$\frac{\|g(x)\|}{\|x\|} \rightarrow 0 \quad (5)$$
Example

Determine whether the following system is almost linear in the neighborhood of the origin.

\[
\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^2 - xy \\ -0.75xy - 0.25y^2 \end{pmatrix}
\]
Linearization

Let us return to the general nonlinear system:

\[ x' = F(x, y), \quad y' = G(x, y) \]  
(6)

Assume that \((x_0, y_0)\) is an isolated critical point.

**Theorem**

*The system (6) is almost linear in the neighborhood of \((x_0, y_0)\) whenever the functions \(F\) and \(G\) have continuous partial derivatives up to order 2.*

**Linearization** using Taylor expansions about \((x_0, y_0)\):

\[
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1 = x - x_0, \ u_2 = y - y_0 \]  
(7)

**Remarks:**

- If \(F\) and \(G\) are twice differentiable, then (6) is almost linear
- (7) is the linear system that corresponds to (6) near \((x_0, y_0)\)
- Matrix in (7) is called Jacobian matrix
Example

Find the linear system corresponding to the pendulum equations near the origin and near the critical point \((\pi, 0)\).

\[
x' = y, \quad y' = -\omega^2 \sin x - \gamma y
\]

Answer:

- **Near origin:**
  \[
  \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
  \]

- **Near \((\pi, 0)\):**
  \[
  \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1 = x - \pi, \ u_2 = y
  \]
Phase Portraits

Consider the almost linear system in the neighborhood of $x = 0$

$$x' = Ax + g(x) \quad (8)$$

Since $g(x) \ll Ax$ when $x$ is small, it is reasonable to expect that the trajectories of the linear system $x' = Ax$ are good approximations of trajectories of (8) near the origin. This is true for many (but not all) cases.

**Theorem**

Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of the linear system $x' = Ax$.

- If $\lambda_{1,2} = \pm i\beta$ (stable center), then the type and stability of $x = 0$ for (8) are
  - Type: Center or Spiral Sink or Spiral Source
  - Stability: Undetermined

- If $\lambda_1 = \lambda_2 > 0$ (unstable degenerate nodal source), then for (8) $x = 0$ is
  - Type: Spiral Source or Nodal Source
  - Stability: Unstable

- If $\lambda_1 = \lambda_2 < 0$ (as. stable degenerate nodal sink), then for (8) $x = 0$ is
  - Type: Spiral Sink or Nodal Sink
  - Stability: Asymptotically Stable

In all other cases, the type and stability of $x = 0$ for the nonlinear system and its linearization are the same.
Summary

- System $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x})$ is called an **almost linear system** in the neighborhood of $\mathbf{x} = \mathbf{0}$ if
  - $\mathbf{g}(\mathbf{x})$ has continuous partial derivatives
  - $\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} \to 0$, as $\mathbf{x} \to \mathbf{0}$

- The system $\mathbf{x}' = F(x, y)$, $\mathbf{y}' = G(x, y)$ is **almost linear** in the neighborhood of $(x_0, y_0)$ whenever the functions $F$ and $G$ are twice differentiable. The corresponding linear system is

$$
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}' =
\begin{pmatrix}
  F_x(x_0, y_0) & F_y(x_0, y_0) \\
  G_x(x_0, y_0) & G_y(x_0, y_0)
\end{pmatrix}
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}, \quad u_1 = x - x_0, \; u_2 = y - y_0
$$

- Relationship between types and stability properties of almost linear systems and their linearizations is given by the theorem on Slide 10.

Homework:

- Section 7.2
  - 5(a,b,c), 7(a,b,c), 13(a,b,c), 19(a,b,c)