Lecture 37. Autonomous Nonlinear Systems and Stability

April 18, 2012
Agenda

- Geometric Approach to analysis of ODEs
- Autonomous Systems
- Existence and Uniqueness of Solutions
- Stability and Instability
- Example: The Oscillating Pendulum
- Drawing nonlinear phase portraits of ODE systems in MATLAB
- Summary and Homework
Geometric Approach

In the next four Lectures, we will discuss systems of nonlinear ODEs.

- Nonlinear systems can rarely be solved by analytical methods.
- In applications, numerical methods are used.
- Geometrical approach provides a qualitative understanding of the behavior of solutions without actually solving the equations.

The numerical and geometrical approaches complement each other:

- the numerical methods provided detailed information about a single solution
- the geometrical methods provide qualitative information about all solutions simultaneously.
Autonomous Systems

We will study two-dimensional autonomous systems of the form

\[
\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y) \quad (1)
\]

- \(F\) and \(G\) are some (nonlinear) functions
- System (1) is called “autonomous” because \(F\) and \(G\) do not depend on the independent variable \(t\).

Let \((x_0, y_0)\) be any point in the \(xy\)-plane.

When does a solution \((x(t), y(t))\) of (1) passing through \((x_0, y_0)\) at time \(t_0\) exist?

Theorem

Suppose that \(F\), \(G\) and their partial derivatives \(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\) are continuous in some domain \(D\) that contains \((x_0, y_0)\). Then there is an interval \(t \in (t_0 - h, t_0 + h)\) in which there exists a unique solution of (1) that also satisfies the initial conditions

\[
\begin{align*}
    x(t_0) &= x_0, \\
    x(y_0) &= y_0
\end{align*} \quad (2)
\]
Stability and Instability

It is convenient to write the initial value problem (1),(2) in the vector form:

\[ \dot{x} = f(x), \quad x(t_0) = x_0 \]  

(3)

- \( x(t) = (x(t), y(t))^T \), \( f(x) = (F(x, y), G(x, y))^T \), \( x_0 = (x_0, y_0) \)
- We interpret a solution \( x = \varphi(t) \) as a curve traced by a moving point in the \( xy \)-plane, the phase plane.

Definition

The points where \( f(x) = 0 \) are called critical points of system (3).

At such points, \( \dot{x} = 0 \), so critical points correspond to equilibrium (constant) solutions of the system. Intuitively:

- A critical point is stable if all trajectories that start close to the critical point remain close to it for all future time.
- A critical point is asymptotically stable if all close trajectories not only remain close but approach the critical point as \( t \to \infty \).
- A critical point is unstable if at least some nearby trajectories do not remain close the critical point as \( t \) increases.
Stability and Instability

\[ x' = f(x), \quad x(t_0) = x_0 \]  

(4)

Definition

- A critical point \( x_0 \) is said to be **stable** if for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that every solution \( x = \varphi(t) \) of the system (4), which at \( t = 0 \) satisfies
  \[ \| \varphi(0) - x_0 \| < \delta, \]  
  \[ (5) \]
  exists for all \( t > 0 \) and satisfies
  \[ \| \varphi(t) - x_0 \| < \varepsilon \]  
  \[ (6) \]

- A critical point that is not stable is said to be **unstable**.

- A critical point \( x_0 \) is said to be **asymptotically stable** if it is stable and if there exists a \( \delta_0 > 0 \) such that, if a solution \( x = \varphi(t) \) satisfies
  \[ \| \varphi(0) - x_0 \| < \delta_0, \]  
  \[ (7) \]
  then
  \[ \lim_{t \to \infty} \varphi(t) = x_0 \]  
  \[ (8) \]
Example: The Oscillating Pendulum

The concepts of stability, asymptotic stability, and instability can easily visualized in terms of an oscillating pendulum.

Recall that (Lecture 15) the equation of motion of an oscillating pendulum is

\[ \theta'' + \gamma \theta' + \omega^2 \sin \theta = 0 \]  \hspace{1cm} (9)

Introducing state variables, \( x = \theta \) and \( y = \theta' \), we convert (9) to a system:

\[ x' = y, \quad y' = -\omega^2 \sin x - \gamma y \]  \hspace{1cm} (10)

The critical points: \( x = \pm n\pi, \ y = 0 \). These points correspond to two physical equilibrium positions:

1. \( \theta = 0 \): the ball is directly below the point of support.
   - If damping coefficient \( \gamma \neq 0 \), then \( (\pm 2\pi n, 0) \) are asymptotically stable
   - If damping coefficient \( \gamma = 0 \), then \( (\pi \pm 2\pi n, 0) \) are stable

2. \( \theta = \pi \): the ball is directly above the point of support — unstable
Drawing Phase Portraits of ODE systems in MATLAB

Usually it is easy to classify critical points from the phase portrait.

Matlab GUI (graphical user interface) PPLANE, written by J. Polking (Rice University), offers a point and click interface for drawing phase portraits of autonomous systems.

- Download: http://math.rice.edu/~dfield/
- Tutorial: http://online.redwoods.cc.ca.us/instruct/darnold/mfiles/review/review.htm

Example: Consider the system

\[ x' = -(2 + y)(x + y), \quad y' = -y(1 - x) \]

- Find all the critical points
- Draw the phase portrait
- For each critical point determine whether it as stable, asymptotically stable, or unstable.
Drawing Phase Portraits of ODE systems in MATLAB

\[ \begin{align*}
    x' &= - (2 + y)(x + y) \\
    y' &= - y (1 - x)
\end{align*} \]

- (1, −2) is asymptotically stable
- (0, 0) is asymptotically stable
- (1, −1) is unstable
Summary

- **Existence and Uniqueness of Solutions:** If $F$, $G$ and $\partial F/\partial x$, $\partial F/\partial y$, $\partial G/\partial x$, $\partial G/\partial y$ are continuous in some domain $D$ that contains $(x_0, y_0)$. Then there is an interval $t \in (t_0 - h, t_0 + h)$ in which there exists a unique solution of

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y)$$

that also satisfies the initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$

- **Stability and Instability:**
  - A critical point is **stable** if all trajectories that start close to the critical point remain close to it for all future time.
  - A critical point is **asymptotically stable** if all close trajectories not only remain close but approach the critical point as $t \to \infty$.
  - A critical point is **unstable** if at least some nearby trajectories do not remain close the critical point as $t$ increases.
Homework

Homework:

- Section 7.1
  - Play with PPLANE: 3, 5, 7, 9, 11