Math 245 - Mathematics of Physics and Engineering I

Lecture 36. Nonhomogeneous Linear Systems

April 16, 2012

Nonhomogeneous Linear Systems: Variation of Parameters

In this Lecture, we finish consideration of linear systems of dimension $n > 2$ by discussing the **method of variation of parameters** for non-homogeneous systems.

Consider the following system:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \tag{1}$$

where the $n \times n$ matrix $\mathbf{P}(t)$ and the $n \times 1$ vector $\mathbf{g}(t)$ are continuous on (α, β). The method of variation of parameters (Lectures 21) can be naturally generalized to n dimensional systems, and a particular solution of (1) can be written as

$$\mathbf{x}_p(t) = \mathbf{X}(t) \int \mathbf{X}^{-1}(t)\mathbf{g}(t)dt, \tag{2}$$

where $\mathbf{X}(t)$ is a fundamental matrix for the corresponding homogeneous system. Alternatively, we can write $\mathbf{x}_p(t)$ as follows:

$$\mathbf{x}_p(t) = \mathbf{X}(t) \int_{t^*}^{t} \mathbf{X}^{-1}(t)\mathbf{g}(t)dt, \tag{3}$$

where $t^* \in (\alpha, \beta)$ is any point.

Nonhomogeneous Linear Systems: Variation of Parameters

The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is then

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int_{t^*}^{t} \mathbf{X}^{-1}(t)\mathbf{g}(t)dt, \tag{4}$$

where $\mathbf{c} = (c_1, c_2)^T$ is a constant vector.

Now let us consider the initial value problem for the n dimensional system:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{5}$$

We can find the solution of (5) most conveniently if we choose $t^* = t_0$.
Then the solution is

$$\boxed{\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(t)\mathbf{g}(t)dt} \tag{6}$$

The Case of Constant Coefficient Matrix: $\mathbf{P}(t) = \mathbf{A}$

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{7}$$

In this case:

- The fundamental matrix
$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} \tag{8}$$
- Its inverse
$$\mathbf{\Phi}^{-1}(t) = e^{-\mathbf{A}t} \tag{9}$$
- The solution of (7)
$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}t}\mathbf{g}(t)dt} \tag{10}$$
- If $t_0 = 0$, then
$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{A}t}\int_{0}^{t} e^{-\mathbf{A}t}\mathbf{g}(t)dt} \tag{11}$$

Example

- Use the method of variation of parameters to find the solution of the IVP:

$$\mathbf{x}' = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sin t \\ t \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Answer:

$$\mathbf{x}(t) = \begin{pmatrix} \frac{7}{2}e^{t} - 3e^{t} + \frac{3}{2}\sin t - \frac{1}{2}\cos t - 4t \\ \frac{7}{2}e^{t} - \frac{3}{2}e^{-t} + \sin t - 3t - 1 \end{pmatrix}$$

Summary

- The solution of the IVP

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is

$$\boxed{\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(t)\mathbf{g}(t)dt}$$

- If $\mathbf{P}(t)$ is a constant matrix, $\mathbf{P}(t) = \mathbf{A}$, then

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}t}\mathbf{g}(t)dt}$$

Homework:

- Section 6.6
  - 1
  - Finish Example on Slide 5