Math 245 - Mathematics of Physics and Engineering I

Lecture 35. Fundamental Matrices and the Exponential of a Matrix - II

April 13, 2012
Agenda

- Properties of $e^{At}$
- Methods for constructing $e^{At}$
- Using the Laplace Transform to Find $e^{At}$
- Summary and Homework
Properties of $e^{At}$

Last time we introduced the matrix exponential function.

**Definition**

Let $A$ be an $n \times n$ constant matrix. The matrix exponential function is defined as follows:

$$e^{At} = I_n + At + \frac{1}{2!}A^2 t^2 + \frac{1}{3!}A^3 t^3 + \ldots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$  \hfill (1)

**Theorem**

Let $A$ and $B$ be $n \times n$ constant matrices, and $t$ and $\tau$ be real or complex numbers. Then

- $e^{A(t+\tau)} = e^{At} e^{A\tau}$
- $A$ commutes with $e^{At}$, that is, $A e^{At} = e^{At} A$
- $(e^{At})^{-1} = e^{-At}$
- If $AB = BA$, then $e^{(A+B)t} = e^{At} e^{Bt}$
Methods for constructing $e^{At}$

**Question:** How to construct $\Phi(t) = e^{At}$? (by definition is difficult!)

Consider the following initial value problem:

$$x' = Ax, \quad x(0) = x_0 \quad (2)$$

Then $\Phi(t) = e^{At}$ is the fundamental matrix, and the solution of (2) is

$$x(t) = e^{At}x_0 \quad (3)$$

On the other hand, if $X(t)$ is any fundamental matrix, then the solution of (2) can be written as follows:

$$x(t) = X(t)X^{-1}(0)x_0 \quad (4)$$

Therefore, from (3), (4), and uniqueness of the solution it follows that

$$e^{At} = X(t)X^{-1}(0) \quad (5)$$
Example

Find $e^{At}$ if

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

Answer:

$$e^{At} = \begin{pmatrix} (1 - t)e^{2t} & -te^{2t} \\ te^{2t} & (1 + t)e^{2t} \end{pmatrix}$$
Methods for constructing $e^{At}$

A special case is when $A$ has $n$ linearly independent eigenvectors $v_1, \ldots, v_n$.

Then a fundamental system for $x' = Ax$ is $\{e^{\lambda_1 t}v_1, \ldots, e^{\lambda_n t}v_n\}$. Thus

$$X(t) = [e^{\lambda_1 t}v_1, \ldots, e^{\lambda_1 t}v_n] = \begin{pmatrix} e^{\lambda_1 t}v_1^1 & \cdots & e^{\lambda_1 t}v_n^1 \\ \vdots & & \vdots \\ e^{\lambda_1 t}v_1^n & \cdots & e^{\lambda_1 t}v_n^n \end{pmatrix} = Ve^{\Lambda t} \quad (6)$$

where

$$V = [v_1, \ldots, v_n], \quad e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \quad (7)$$

Therefore, in this case:

$$e^{At} = Ve^{\Lambda t}V^{-1} \quad (8)$$
Using the Laplace Transform to Find $e^{At}$

Consider the following matrix initial value problem:

$$\Phi' = A\Phi, \quad \Phi(0) = I_n$$

(9)

- (9) is equivalent to $n$ initial value problems for the individual columns of $\Phi$.
- $e^{At}$ is the solution.

Taking the Laplace transform of the ODE in (9) and solving to $\mathcal{L}\{\Phi\}$ yields

$$\mathcal{L}\{\Phi\} = (sI_n - A)^{-1}$$

(10)

We can then recover $\Phi$ by taking the inverse Laplace transform. Thus

$$e^{At} = \mathcal{L}^{-1}\{(sI_n - A)^{-1}\}$$

(11)
Example

Find $e^{At}$ if

$$A = \begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}$$

Answer:

$$e^{At} = \begin{pmatrix} e^t \cos t - 2e^t \sin t & -5e^t \sin t \\ e^t \sin t & e^t \cos t + 2e^t \sin t \end{pmatrix}$$
Summary

- Properties of $e^{At}$:
  - $e^{A(t+\tau)} = e^{At} e^{A\tau}$
  - $A$ commutes with $e^{At}$, that is, $Ae^{At} = e^{At}A$
  - $(e^{At})^{-1} = e^{-At}$
  - If $AB = BA$, then $e^{(A+B)t} = e^{At} e^{Bt}$

- Methods for constructing $e^{At}$:
  - If $X(t)$ is any fundamental matrix for $x' = Ax$, then
    $$e^{At} = X(t)X^{-1}(0)$$
  - If $A$ has $n$ linearly independent eigenvectors, then
    $$e^{At} = V e^{A't} V^{-1}$$
  - Using the inverse Laplace transform:
    $$e^{At} = \mathcal{L}^{-1}\left\{ (sI_n - A)^{-1} \right\}$$
Homework

Homework:

- Section 6.5
  - 9, 11, 17