Math 245 - Mathematics of Physics and Engineering I

Lecture 35. Fundamental Matrices and the Exponential of a Matrix - II

April 13, 2012

Agenda

- Properties of $e^{\mathbf{A}t}$
- Methods for constructing $e^{\mathbf{A}t}$
- Using the Laplace Transform to Find $e^{\mathbf{A}t}$
- Summary and Homework

Properties of $e^{\mathbf{A}t}$

Last time we introduced the matrix exponential function.

Definition

Let $\mathbf{A}$ be an $n \times n$ constant matrix. The matrix exponential function is defined as follows:

$$e^{\mathbf{A}t} = \mathbf{I}_n + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \frac{1}{3!}\mathbf{A}^3 t^3 + \ldots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} \tag{1}$$

Theorem

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ constant matrices, and t and τ be real or complex numbers. Then

- $e^{\mathbf{A}(t+\tau)} = e^{\mathbf{A}t} e^{\mathbf{A}\tau}$
- $\mathbf{A}$ *commutes with* $e^{\mathbf{A}t}$, *that is,* $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$
- $(e^{\mathbf{A}t})^{-1} = e^{-\mathbf{A}t}$
- *If* $\mathbf{AB} = \mathbf{BA}$, *then* $e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t} e^{\mathbf{B}t}$

Methods for constructing $e^{\mathbf{A}t}$

Question: How to construct $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$? (by definition is difficult!)

Consider the following initial value problem:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2}$$

Then $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ is the fundamental matrix, and the solution of (2) is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 \tag{3}$$

On the other hand, if $\mathbf{X}(t)$ is any fundamental matrix, then the solution of (2) can be written as follows:

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(0)\mathbf{x}_0 \tag{4}$$

Therefore, from (3), (4), and uniqueness of the solution it follows that

$$\boxed{e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0)} \tag{5}$$

Example

- Find $e^{\mathbf{A}t}$ if

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

<u>Answer:</u>

$$e^{\mathbf{A}t} = \begin{pmatrix} (1-t)e^{2t} & -te^{2t} \\ te^{2t} & (1+t)e^{2t} \end{pmatrix}$$

Methods for constructing $e^{\mathbf{A}t}$

A special case is when $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Then a fundamental system for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is $\{e^{\lambda_1 t}\mathbf{v}_1, \ldots, e^{\lambda_n t}\mathbf{v}_n\}$.Thus

$$\mathbf{X}(t) = [e^{\lambda_1 t}\mathbf{v}_1, \ldots, e^{\lambda_1 t}\mathbf{v}_n] = \begin{pmatrix} e^{\lambda_1 t} v_1^1 & \ldots & e^{\lambda_n t} v_n^1 \\ \vdots & & \vdots \\ e^{\lambda_1 t} v_1^n & \ldots & e^{\lambda_n t} v_n^n \end{pmatrix} = \mathbf{V} e^{\mathbf{\Lambda} t} \tag{6}$$

where

$$\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n], \qquad e^{\mathbf{\Lambda} t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix} \tag{7}$$

Therefore, in this case:

$$\boxed{e^{\mathbf{A}t} = \underbrace{\mathbf{V} e^{\mathbf{\Lambda} t}}_{\mathbf{X}(t)} \underbrace{\mathbf{V}^{-1}}_{\mathbf{X}^{-1}(0)}} \tag{8}$$

Using the Laplace Transform to Find $e^{\mathbf{A}t}$

Consider the following matrix initial value problem:

$$\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}, \qquad \mathbf{\Phi}(0) = \mathbf{I}_n \tag{9}$$

- (9) is equivalent to n initial value problems for the individual columns of $\mathbf{\Phi}$.
- $e^{\mathbf{A}t}$ is the solution.

Taking the Laplace transform of the ODE in (9) and solving to $\mathcal{L}\{\mathbf{\Phi}\}$ yields

$$\mathcal{L}\{\mathbf{\Phi}\} = (s\mathbf{I}_n - \mathbf{A})^{-1} \tag{10}$$

We can then recover $\mathbf{\Phi}$ by taking the inverse Laplace transform. Thus

$$\boxed{e^{\mathbf{A}t} = \mathcal{L}^{-1}\left\{(s\mathbf{I}_n - \mathbf{A})^{-1}\right\}} \tag{11}$$

Example

- Find $e^{\mathbf{A}t}$ if

$$\mathbf{A} = \begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}$$

Answer:

$$e^{\mathbf{A}t} = \begin{pmatrix} e^t \cos t - 2e^t \sin t & -5e^t \sin t \\ e^t \sin t & e^t \cos t + 2e^t \sin t \end{pmatrix}$$

Summary

- Properties of $e^{\mathbf{A}t}$:
  - $e^{\mathbf{A}(t+\tau)} = e^{\mathbf{A}t}e^{\mathbf{A}\tau}$
  - $\mathbf{A}$ commutes with $e^{\mathbf{A}t}$, that is, $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$
  - $(e^{\mathbf{A}t})^{-1} = e^{-\mathbf{A}t}$
  - If $\mathbf{AB} = \mathbf{BA}$, then $e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t}e^{\mathbf{B}t}$

- Methods for constructing $e^{\mathbf{A}t}$:
  - If $\mathbf{X}(t)$ is any fundamental matrix for $\mathbf{x}' = \mathbf{Ax}$, then

$$e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$$

  - If $\mathbf{A}$ has n linearly independent eigenvectors, then

$$e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}$$

  - Using the inverse Laplace transform:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left\{(s\mathbf{I}_n - \mathbf{A})^{-1}\right\}$$

Homework

Homework:

- Section 6.5
  - 9, 11, 17