*Math 245 - Mathematics of Physics and Engineering I*

# Lecture 34. Fundamental Matrices and the Exponential of a Matrix - I

April 11, 2012

# Fundamental Matrix

In this Lecture, the goal is to describe the structure of the solutions of the general homogeneous system of linear first order ODEs:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{1}$$

Suppose $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ form a fundamental set of solutions for (1) on some interval $t \in (\alpha, \beta)$. Then the fundamental matrix is

$$\mathbf{X}(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)] = \begin{pmatrix} x_1^1(t) & \ldots & x_n^1(t) \\ \vdots & \ddots & \vdots \\ x_1^n(t) & \ldots & x_n^n(t) \end{pmatrix} \tag{2}$$

- $\mathbf{X}(t)$ is nonsingular ($\det \mathbf{X}(t) \neq 0$ for $t \in (\alpha, \beta)$), since its columns are linearly independent vectors.

The general solution of (1) is then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = \mathbf{X}(t)\mathbf{c}, \tag{3}$$

where $\mathbf{c} = (c_1, \ldots, c_n)^T$.

# Fundamental Matrix $\Phi(t)$

If we have an IVP:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t_0 \in (\alpha, \beta) \tag{4}$$

then $\mathbf{c}$ must satisfy

$$\mathbf{X}(t_0)\mathbf{c} = \mathbf{x}_0 \quad \Rightarrow \quad \mathbf{c} = \mathbf{X}^{-1}(t_0)\mathbf{x}_0 \tag{5}$$

Therefore, the solution of (4) is given by

$$\boxed{\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0} \tag{6}$$

Let $\mathbf{\Phi}(t)$ be the special fundamental matrix whose columns are the vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ that are solutions of (4) with the initial conditions

$$\mathbf{x}_1(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{x}_2(t_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \ldots \quad \mathbf{x}_n(t_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{7}$$

# Fundamental Matrix $\Phi(t)$

- Example: Find the fundamental matrix $\mathbf{\Phi}(t)$ for the following system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}, \quad t_0 = 0$$

Answer:

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{3t}/2 + e^{-t}/2 & e^{3t}/4 - e^{-t}/4 \\ e^{3t} - e^{-t} & e^{3t}/2 + e^{-t}/2 \end{pmatrix}$$

Fundamental matrix $\mathbf{\Phi}(t)$ has the following property:

$$\mathbf{\Phi}(t_0) = \mathbf{I}_n \tag{8}$$

Thus, in terms of $\mathbf{\Phi}(t)$, the solution of the initial value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{9}$$

is

$$\boxed{\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}_0} \tag{10}$$

This, if we know $\mathbf{\Phi}(t)$, then it is very easy to solve the IVP (9) for any initial condition $\mathbf{x}_0$: just use (10).

# The Matrix Exponential Function

<u>Motivation:</u> Let us compare the following two observations:

- The solution of the IVP $x' = ax$, $x(0) = x_0$ is

$$\boxed{x(t) = e^{at}x_0} \tag{11}$$

- The solution of the IVP $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\boxed{\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}_0} \tag{12}$$

Comparing the problems and solutions (11) and (12), suggests that

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t},$$

whatever the last equation means...

# The Matrix Exponential Function

Recall that the scalar exponential function can be represented by the power series:

$$e^{at} = 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \ldots = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} \tag{13}$$

## Definition

Let $\mathbf{A}$ be an $n \times n$ constant matrix. The **matrix exponential function** is defined as follows:

$$\boxed{e^{\mathbf{A}t} = \mathbf{I}_n + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \ldots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}} \tag{14}$$

- $\mathbf{A}^k = \mathbf{A} \times \mathbf{A} \times \ldots \times \mathbf{A}$ ($k$ times)
- More accurately, $e^{\mathbf{A}t} = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{\mathbf{A}^k t^k}{k!}$
- It can be shown that the above sum indeed converges (quite rapidly), and the limit matrix is denoted by $e^{\mathbf{A}t}$.

# Example

- Find $e^{\mathbf{A}t}$ if

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

<u>Answer:</u>

$$e^{\mathbf{A}t} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

In general, it is not possible to express the entries of $e^{\mathbf{A}t}$ in terms of elementary functions. But if $\mathbf{A}$ is diagonal, then it is easy to do:

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \quad \Rightarrow \quad e^{\mathbf{A}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}$$

# $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$

The following theorem shows the equivalence between $e^{\mathbf{A}t}$ and $\mathbf{\Phi}(t)$.

## Theorem

*Consider the following IVP* $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$. *Then*

$$\boxed{\mathbf{\Phi}(t) = e^{\mathbf{A}t}}$$

*Therefore, the solutions of the IVP is*

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

# Summary

- If $\mathbf{X}(t)$ is any fundamental matrix for system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, then the solution of the IVP $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\boxed{\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0}$$

- If $\mathbf{\Phi}(t)$ is the special fundamental matrix (see Slide 4), then the solution of the IVP can be written as follows:

$$\boxed{\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}_0}$$

- Consider the system with constant coefficients: $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $t_0 = 0$. Then
  - $\boxed{\mathbf{\Phi}(t) = e^{\mathbf{A}t}}$ where
  - $\boxed{e^{\mathbf{A}t} = \mathbf{I}_n + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \ldots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}}$

# Homework

Homework:

- Section 6.5
  - 3, 5, 7