Math 245 - Mathematics of Physics and Engineering I

Lecture 33. Homogeneous Linear Systems with Constant Coefficients: Complex Eigenvalues

April 9, 2012

Complex Eigenvalues

In this Lecture, we study systems of linear homogeneous ODEs

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where the coefficient matrix $\mathbf{A}$ has a complete set of n linearly independent eigenvectors and one or more pairs of complex conjugate eigenvalues.

Since $\mathbf{A}$ is real, the coefficients of the characteristic equation are real.

$$\det(\mathbf{A} - \lambda \mathbf{I}_n) = 0$$

Therefore, complex eigenvalues must occur in conjugate pairs:

- If $\lambda = \alpha + i\beta$ is an eigenvalue, then so is $\bar{\lambda} = \alpha - i\beta$.
- If $\mathbf{v}$ is an eigenvector corresponding to λ, then $\bar{\mathbf{v}}$ is an eigenvalue corresponding to $\bar{\lambda}$.

Thus, the following complex conjugate vector functions are solutions of (1):

$$\mathbf{u}(t) = e^{\lambda t}\mathbf{v}, \qquad \bar{\mathbf{u}}(t) = e^{\bar{\lambda} t}\bar{\mathbf{v}} \tag{2}$$

We obtain real solutions of (1) by taking the real and imaginary parts of $\mathbf{u}(t)$:

$$\mathbf{x}_1(t) = \mathrm{Re}(\mathbf{u}(t)), \qquad \mathbf{x}_2(t) = \mathrm{Im}(\mathbf{u}(t)) \tag{3}$$

Complex Eigenvalues

Real solutions can be written in the following form:

$$\boxed{\begin{aligned} \mathbf{x}_1(t) &= \mathrm{Re}(\mathbf{u}(t)) = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) \\ \mathbf{x}_2(t) &= \mathrm{Im}(\mathbf{u}(t)) = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t) \end{aligned}} \tag{4}$$

where

$$\mathbf{a} = \mathrm{Re}(\mathbf{v}), \qquad \mathbf{b} = \mathrm{Im}(\mathbf{v}), \qquad \mathbf{v} = \mathbf{a} + i\mathbf{b} \tag{5}$$

- It can be shown that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent.

If all eigenvectors of $\mathbf{A}$ (real and complex) are linearly independent, then a fundamental set of real solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ consists of

- solutions of form (4) associated with complex eigenvalues
- solutions of form $e^{\lambda t}\mathbf{v}$ associated with real eigenvalues

Example

Find the general solution of the following system of ODEs in terms of real-valued functions:

$$\mathbf{x}' = \begin{pmatrix} 0 & -2 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & -2 \end{pmatrix} \mathbf{x}$$

<u>Answer:</u>

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \\ \cos 2t \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} \sin 2t \\ -\cos 2t \\ \sin 2t \end{pmatrix}$$

Summary

- If all eigenvectors of $\mathbf{A}$ (real and complex) are linearly independent, then a fundamental set of real solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ consists of
  - solutions associated with complex eigenvalues $\lambda = \alpha \pm i\beta$:

$$\begin{aligned} \mathbf{x}_1(t) &= \mathrm{Re}(\mathbf{u}(t)) = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) \\ \mathbf{x}_2(t) &= \mathrm{Im}(\mathbf{u}(t)) = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t) \end{aligned} \tag{6}$$

  - solutions of form $e^{\lambda t}\mathbf{v}$ associated with real eigenvalues λ

Homework:

- Section 6.4
  - 1, 5, 7