Lecture 33. Homogeneous Linear Systems with Constant Coefficients: Complex Eigenvalues
Complex Eigenvalues

In this Lecture, we study systems of linear homogeneous ODEs

$$x' = Ax,$$  \hspace{0.5cm} (1)

where the coefficient matrix $A$ has a complete set of $n$ linearly independent eigenvectors and one or more pairs of complex conjugate eigenvalues.

Since $A$ is real, the coefficients of the characteristic equation are real.

$$\det(A - \lambda I_n) = 0$$

Therefore, complex eigenvalues must occur in conjugate pairs:

- If $\lambda = \alpha + i\beta$ is an eigenvalue, then so is $\bar{\lambda} = \alpha - i\beta$.
- If $v$ is an eigenvector corresponding to $\lambda$, then $\bar{v}$ is an eigenvalue corresponding to $\bar{\lambda}$.

Thus, the following complex conjugate vector functions are solutions of (1):

$$u(t) = e^{\lambda t} v, \quad \bar{u}(t) = e^{\bar{\lambda} t} \bar{v}$$ \hspace{0.5cm} (2)

We obtain real solutions of (1) by taking the real and imaginary parts of $u(t)$:

$$x_1(t) = \text{Re}(u(t)), \quad x_2(t) = \text{Im}(u(t))$$ \hspace{0.5cm} (3)
Complex Eigenvalues

Real solutions can be written in the following form:

\[
\begin{align*}
x_1(t) &= \text{Re}(u(t)) = e^{\alpha t}(a \cos \beta t - b \sin \beta t) \\
x_2(t) &= \text{Im}(u(t)) = e^{\alpha t}(a \sin \beta t + b \cos \beta t)
\end{align*}
\] (4)

where

\[
a = \text{Re}(v), \quad b = \text{Im}(v), \quad v = a + ib
\] (5)

- It can be shown that \(x_1(t)\) and \(x_2(t)\) are linearly independent.

If all eigenvectors of \(A\) (real and complex) are linearly independent, then a fundamental set of real solutions of \(x' = Ax\) consists of

- solutions of form (4) associated with complex eigenvalues
- solutions of form \(e^{\lambda t}v\) associated with real eigenvalues
Example

Find the general solution of the following system of ODEs in terms of real-valued functions:

\[
x' = \begin{pmatrix} 0 & -2 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & -2 \end{pmatrix} x
\]

Answer:

\[
x(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \\ \cos 2t \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} \sin 2t \\ -\cos 2t \\ \sin 2t \end{pmatrix}
\]
Summary

- If all eigenvectors of $\mathbf{A}$ (real and complex) are linearly independent, then a fundamental set of real solutions of $\mathbf{x}' = \mathbf{Ax}$ consists of
  - solutions associated with complex eigenvalues $\lambda = \alpha \pm i\beta$:
    
    \begin{align*}
    x_1(t) &= \text{Re}(u(t)) = e^{\alpha t}(a \cos \beta t - b \sin \beta t) \\
    x_2(t) &= \text{Im}(u(t)) = e^{\alpha t}(a \sin \beta t + b \cos \beta t)
    \end{align*}

  - solutions of form $e^{\lambda t} \mathbf{v}$ associated with real eigenvalues $\lambda$

Homework:

- Section 6.4
  - 1, 5, 7