Math 245 - Mathematics of Physics and Engineering I

Lecture 32. Homogeneous Linear Systems with Constant Coefficients

April 6, 2012

Generalization of the Eigenvalue Method

In Lectures 10-13, we discussed the eigenvalue method for solving linear systems with constants coefficients of dimension 2:

$$\mathbf{x}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{x}$$

<u>Goal:</u> to extend the eigenvalue method to systems of dimension $n > 2$

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{1}$$

As before, we look for a solution of (1) in the following form:

$$\mathbf{x} = e^{\lambda t}\mathbf{v} \tag{2}$$

- λ is the scalar,
- $\mathbf{v}$ is the constant $n \times 1$ vector.

This leads to the eigenvalue problem for $\mathbf{A}$:

$$(\mathbf{A} - \lambda \mathbf{I}_n)\mathbf{v} = 0 \tag{3}$$

Three Cases

When solving the eigenvalue problem

$$(\mathbf{A} - \lambda \mathbf{I}_n)\mathbf{v} = 0$$

there are 3 possible cases:

1. $\mathbf{A}$ has a complete set of n linearly independent eigenvectors and all eigenvalues are real,
2. $\mathbf{A}$ has a complete set of n linearly independent eigenvectors and one or more pairs of complex conjugate eigenvalues,
3. $\mathbf{A}$ is defective, i.e. there are one or more eigenvalues of $\mathbf{A}$ for which the geometric multiplicity is less than the algebraic multiplicity (out of our scope)

In this Lecture, we focus of Case 1.

Fundamental System of Solutions

The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is described by the following theorem

Theorem

Let $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$ *be eigenpairs for the real* $n \times n$ *matrix* $\mathbf{A}$*. Assume that*

- *the eigenvalues* $\lambda_1, \ldots, \lambda_n$ *are real,*
- *the eigenvectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are linearly independent*

Then

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \ldots, \mathbf{x}_n = e^{\lambda_n t}\mathbf{v}_n$$

is a fundamental set of solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *on the interval* $(-\infty, \infty)$*. The general solution of* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *is then given by*

$$x(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n$$

Remark: In this theorem, the eigenvalues need not be distinct.

Real and Distinct Eigenvalues

A frequently occurring special case for which the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is always of the form

$$x(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n$$

is when all eigenvalues are distinct.

Theorem

Suppose that the matrix $\mathbf{A}$ *has* n *eigenpairs* $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$ *with the property that the eigenvalues* $\lambda_1, \ldots, \lambda_n$ *are real and distinct. Then*

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \ldots, \mathbf{x}_n = e^{\lambda_n t}\mathbf{v}_n$$

is a fundamental set of solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *on the interval* $(-\infty, \infty)$.

Symmetric Matrices

Recall that matrix $\mathbf{A}$ is called symmetric if $\mathbf{A} = \mathbf{A}^T$, i.e. $a_{ij} = a_{ji}$.

Eigenvalues and eigenvectors of symmetric matrices have the following useful properties:

- All eigenvalues are real.
- There always exists a complete set of n linearly independent eigenvectors.
- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors that correspond to different eigenvalues, then

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$$

- Moreover, all eigenvectors belonging to the same eigenvalue can be chosen to be orthogonal to one another.

Therefore, if A is symmetric, then $\mathbf{x}' = \mathbf{A}\mathbf{x}$ will always have a fundamental set of solutions of the form

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \dots, \mathbf{x}_n = e^{\lambda_n t}\mathbf{v}_n$$

Example: Diffusion of a One-Dimensional Lattice

Consider the following continuous time, discrete space model of particle diffusion.

Suppose that particles can occupy any of n equally spaced sites lying along the real line. Let $x_i(t)$ be the number of particles residing at the i^{th} site at time t. Assume that

- particle transition to site i are permitted only from the nearest-neighbor sites
- particles move from more populated sites to less populated sites
- the rate of transition is proportional to the difference between the numbers of particles at adjacent sites.

Then the differential equation describing the rate of change in the number of particles at site i is:

- If $i = 2, \ldots, n-1$

$$x_i' = k(x_{j-1} - 2x_j + x_{j+1})$$

- If $i = 1$

$$x_1' = k(x_2 - x_1)$$

- If $i = n$

$$x_n' = k(x_{n-1} - x_n)$$

Example: Diffusion of a One-Dimensional Lattice

Consider a special case:

- $k = 1$
- $n = 3$

Then

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{x}$$

The general solution is

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Therefore

$$\lim_{t \to \infty} \mathbf{x}(t) = \begin{pmatrix} c_1 \\ c_1 \\ c_1 \end{pmatrix}$$

That is, all solutions approach an equilibrium state in which the numbers of particles at each site are identical (uniform distribution).

Summary

- Let $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$ be eigenpairs for the real $n \times n$ matrix $\mathbf{A}$. Assume that
  - the eigenvalues $\lambda_1, \ldots, \lambda_n$ are real,
  - the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent

  Then

  $$\boxed{\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \ldots, \mathbf{x}_n = e^{\lambda_n t}\mathbf{v}_n}$$

  is a fundamental set of solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on the interval $(-\infty, \infty)$.
- Suppose that eigenvalues $\lambda_1, \ldots, \lambda_n$ of matrix $\mathbf{A}$ are real and distinct. Then

  $$\boxed{\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \ldots, \mathbf{x}_n = e^{\lambda_n t}\mathbf{v}_n}$$

  is a fundamental set of solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on the interval $(-\infty, \infty)$.
- Symmetric Matrices $\mathbf{A} = \mathbf{A}^T$:
  - All eigenvalues are real.
  - There always exists a complete set of n linearly independent eigenvectors.
  - $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \ldots, \mathbf{x}_n = e^{\lambda_n t}\mathbf{v}_n$ is a fundamental set of solutions.

Homework

Homework:

- Section 6.3
  - 3, 7
  - Solve the IVP: 9