Math 245 - Mathematics of Physics and Engineering I

Lecture 31. Basic Theory of First Order Linear Systems

April 4, 2012

Agenda

- Existence and Uniqueness
- Properties and Structure of Solutions of Homogeneous Systems
  - Principle of Superposition
  - Linear Independence of Solutions
  - Wronskian
  - Existence of a Fundamental Set of Solutions
- Linear n^{th} order ODEs
- Homework

Existence and Uniqueness

The general first order linear system of dimension n has the following form:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t) \tag{1}$$

where $\mathbf{P}(t)$ is an $n \times n$ matrix and $\mathbf{g}(t)$ a $n \times 1$ vector.

Theorem

If $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on an open interval $I = (\alpha, \beta)$, then there exists a unique solution $\mathbf{x} = \mathbf{z}(t)$ of the initial value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where $t_0 \in I$, and $\mathbf{x}_0$ is any constant vector with n components. Moreover, the solution exists throughout the interval I.

Important Special Case: Consider the following IVP:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{3}$$

where $\mathbf{A}$ is a constant $n \times n$ matrix. The above theorem guarantees that a solution exists and is unique on the entire t-axis.

Principle of Superposition

Consider the homogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{4}$$

Definition

If $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are solutions of system (4), then an expression of the form

$$c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k, \tag{5}$$

where $c_1, \ldots, c_k$ are arbitrary constants, is called a **linear combination** of solutions.

Principle of Superposition

If $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are solutions of system (4) on the interval I, then any linear combination $c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k$ is also a solution of (4) on I.

- Using the Principle of Superposition, we can enlarge a finite set of solutions $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ to a k-dimensional infinite family of solutions $c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k$ parameterized by $c_1, \ldots, c_k$.

Linear (In)Dependence

Goal: to show that all solutions of n-dimensional system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ are contained in an n-parameter family $c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n$, provided that the n solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are distinct (=linearly independent).

Definition

- The n vector functions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are said to be **linearly independent on an interval** I if the only constants $c_1, \ldots, c_n$ such that

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = 0 \tag{6}$$

  for all $t \in I$ are $c_1 = c_2 = \ldots = c_n = 0$.
- If there exist constants $c_1, \ldots, c_n$, not all zero, such that (6) is true for all $t \in I$, the vector functions are said to be **linearly dependent** on I.

Example: Show that the following vector functions are linearly independent on $I = (-\infty, \infty)$

$$\mathbf{x}_1(t) = \begin{pmatrix} e^{-2t} \\ 0 \\ -e^{-2t} \end{pmatrix} \qquad \mathbf{x}_2(t) = \begin{pmatrix} e^{t} \\ e^{t} \\ e^{t} \end{pmatrix}$$

Wronskian

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ and let $\mathbf{X}(t)$ be $n \times n$ matrix whose j^{th} column is $\mathbf{x}_j(t)$, $j = 1, \ldots, n$.

Definition

The Wronskian $W = W[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ of the n solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is defined by

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det \mathbf{X}(t) \tag{7}$$

Theorem

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an interval $I = (\alpha, \beta)$ in which $\mathbf{P}(t)$ is continuous.

- *If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent on I, then $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) \neq 0$ at every point in I*
- *If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent on I, then $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = 0$ at every point in I*

Structure of Solutions

The following theorem shows that all solutions of an n-dimensional homogeneous system are contained in the n-parameter infinite family of solutions, provided that these solutions are linearly independent.

Theorem

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *be solutions of*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{8}$$

on an interval $I = (\alpha, \beta)$ *such that, for some point* $t_0 \in I$*, the Wronskian is nonzero,* $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t_0) \neq 0$*. Then each solution* $\mathbf{x} = \mathbf{z}(t)$ *of (8) can be written as a linear combination of* $\mathbf{x}_1, \ldots, \mathbf{x}_n$*,*

$$\mathbf{z}(t) = \hat{c}_1\mathbf{x}_1(t) + \ldots + \hat{c}_n\mathbf{x}_n(t), \tag{9}$$

where the constants $\hat{c}_1, \ldots, \hat{c}_n$ *are uniquely defined.*

Remark: If $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t_0) \neq 0$, then

- $c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t)$ is called the general solution.
- $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$ is called a fundamental set of solutions.

Existence of a Fundamental Set of Solutions

Example: Let

$$A = \begin{pmatrix} 0 & -1 & 2 \\ 2 & -3 & 2 \\ 3 & -3 & 1 \end{pmatrix}$$

show that the following solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ form a fundamental set

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{x}_3(t) = e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

It turns out that $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ always has a fundamental set of solutions.

Theorem

Let $\mathbf{e}_1 = (1, 0, \ldots, 0)^T$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)^T$, …, $\mathbf{e}_n = (0, \ldots, 0, 1)^T$.
Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *be solutions of* $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ *that satisfy the initial conditions*

$$\mathbf{x}_1(t_0) = \mathbf{e}_1, \ldots, \mathbf{x}_n(t_0) = \mathbf{e}_n, \quad t_0 \in I$$

Then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *form a fundamental set of solutions.*

Linear n^{th} order ODEs

The IVP for the linear n^{th} order ODE is given by

$$\begin{cases} y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = g(t), \\ y(t_0) = y_0, \;\; y'(t_0) = y_1, \;\; \ldots \;\; , \;\; y^{(n-1)}(t_0) = y_{n-1}. \end{cases} \tag{10}$$

Corollary

If the functions $p_1(t), \ldots, p_n(t)$, and $g(t)$ are continuous on the open interval $I = (\alpha, \beta)$, then there exists exactly one solution $y = z(t)$ of the initial value problem (10). This solution exists throughout the interval I.

Corollary

Let $y_1, \ldots, y_n$ be solutions of

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = 0 \tag{11}$$

on I in which $p_1(t), \ldots, p_n(t)$ are continuous. If for some point $t_0 \in I$, the Wronskian $W[y_1, \ldots, y_n] \neq 0$, then any solution $y = z(t)$ of (11) can be written as a linear combination of $y_1, \ldots, y_n$, $z(t) = \hat{c}_1 y_1(t) + \ldots + \hat{c}_n y_n(t)$, where the constants $\hat{c}_1, \ldots, \hat{c}_n$ are uniquely determined.

Homework

Homework:

- Section 6.2
  - 5, 9, 11