*Math 245 - Mathematics of Physics and Engineering I*

# Lecture 30. Systems of First Order Linear ODEs: Definitions and Examples

April 2, 2012

# Agenda

- General Framework
- Linear $n^{\mathrm{th}}$ order ODEs
- Applications Modeled by First Order Linear Systems
  - Coupled Mass-Spring Systems
  - Linear Control Systems
- Summary and Homework

# General Framework

Our next goal is to develop an elementary theory and solution techniques for **first order linear systems**. We have already discussed the two-dimensional case

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + g_1(t), \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + g_2(t). \end{cases}$$

Now, the goal is investigate the general first order system of $n$ dimensions:

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t), \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t), \\ \ldots\ldots\ldots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t), \end{cases}$$

or, using matrix notation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \ldots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \ldots & p_{nn}(t) \end{pmatrix} \qquad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

# General Framework

- $\mathbf{P}(t)$ is referred to as the matrix of coefficients of the system
- $\mathbf{g}(t)$ is referred to as the nonhomogeneous term of the system
  - if $\mathbf{g}(t) = 0$, the the system is called homogeneous
  - if $\mathbf{g}(t) \neq 0$, the the system is called nonhomogeneous

## Definition

The system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is said to have a solution on the interval $I = (\alpha, \beta)$, if there exists a vector $\mathbf{x} = \mathbf{z}(t)$ with $n$ components that is differentiable at all points in the interval $I$ and satisfies $\mathbf{z}(t)' = \mathbf{P}(t)\mathbf{z}(t) + \mathbf{g}(t)$

If in addition to the system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t) \tag{1}$$

an initial condition is given

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{2}$$

then (1) and (2) form an initial value problem.

# Linear $n^{\text{th}}$ order ODEs

Single linear ODEs of higher order can always be transformed into systems of first order linear equations. An $n^{\text{th}}$ order linear ODE in the standard form is given by

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{3}$$

To transform (3) into a system of $n$ first order ODEs, introduce the state variables:

$$x_1 = y, \;\; x_2 = y', \;\; x_3 = y'', \;\; \ldots, \;\; x_n = y^{(n-1)} \tag{4}$$

It then follows immediately that

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_{n-1}' \\ x_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -p_n(t) & -p_{n-1}(t) & -p_{n-2}(t) & \ldots & -p_1(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}$$

# Example: Coupled Mass-Spring Systems

Consider two masses $m_1$ and $m_2$, connected to three springs, and assume that

1. The masses are constrained to move only in the horizontal direction on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$
2. The springs obey Hooke's law, have spring constants $k_1, k_2$, and $k_3$, and, when system is at equilibrium, we assume that the springs are at their rest lengths.

Denote the displacements of $m_1$ and $m_2$ from their equilibrium positions by $y_1$ and $y_2$, respectively. Then the behavior of the system is described by the following system:

$$\begin{cases} m_1 \dfrac{d^2 y_1}{dt^2} = -(k_1 + k_2) y_1 + k_2 y_2 + F_1(t), \\ m_2 \dfrac{d^2 y_2}{dt^2} = -(k_2 + k_3) y_2 + k_2 y_1 + F_2(t). \end{cases} \tag{5}$$

We can transform (5) to a system of linear ODEs by introducing state variables:

$$x_1 = y_1, \;\; x_2 = y_2, \;\; x_3 = y_1', \;\; x_4 = y_2' \tag{6}$$

# Example: Coupled Mass-Spring Systems

Then

$$\begin{cases} x_1' = x_3 \\ x_2' = x_4 \\ x_3' = -\dfrac{k_1 + k_2}{m_1} x_1 + \dfrac{k_2}{m_1} x_2 + \dfrac{1}{m_1} F_1(t) \\ x_4' = \dfrac{k_2}{m_2} x_1 - \dfrac{k_2 + k_3}{m_2} x_2 + \dfrac{1}{m_2} F_2(t) \end{cases} \tag{7}$$

or, using matrix notation,

$$\mathbf{x}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ \frac{F_1(t)}{m_1} \\ \frac{F_2(t)}{m_2} \end{pmatrix} \tag{8}$$

# Example: Linear Control Systems

There are many physical, biological, and engineering systems in which it is desirable to **control** the state of the system. A standard mathematical model for linear control systems consists of the pair of equations:

$$\begin{cases} \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{9}$$

where

- $\mathbf{A}$ is an $n \times n$ system matrix
- $\mathbf{B}$ is an $n \times m$ input matrix
- $\mathbf{C}$ is an $r \times n$ output matrix
- The first and the second equations are referred to as the plant equation and the output equation, respectively.
- $\mathbf{x}$ is state, $\mathbf{y}$ is output, and $\mathbf{u}$ is the plant input.

A common type of control problems is to choose or design the input $\mathbf{u}(t)$ in order to achieve some desired output $\mathbf{y}(t)$.

# Homework

Homework:

- Section 6.1
  - 2, 5, 9