Math 245 - Mathematics of Physics and Engineering I

Lecture 29. Convolution Integrals and Their Applications

March 30, 2012

Agenda

- Convolution
  - Motivation
  - Definition
  - Properties
- The Convolution Theorem
- Input-Output Problems
- Summary and Homework

Motivation and Definition of Convolution

Consider the initial value problem

$$y'' + y = g(t), \quad y(0) = 0, \quad y'(0) = 0,$$

It is easy to check that the following function is the solution

$$y(t) = \int_0^t \sin(t - \tau) g(\tau) d\tau$$

The integral that appears on the r.h.s. is called a convolution integral.

Definition

Let $f(t)$ and $g(t)$ be piecewise continuous functions on $[0, \infty)$.
The **convolution of** f **and** g is defined by

$$\boxed{(f \star g)(t) = \int_0^t f(t - \tau) g(\tau) d\tau} \tag{1}$$

Convolution integrals arise often in representing the output $y(t)$ of a linear ODE with constant coefficients to an input $g(t)$ in the $t-$domain.

Properties of Convolution

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

The notation $f \star g$ is used to emphasize that the convolution has several properties of ordinary multiplication, and $f \star g$ can be considered as a "generalized product".

Theorem

- $f \star g = g \star f$
- $f \star (g_1 + g_2) = f \star g_1 + f \star g_2$
- $(f \star g) \star h = f \star (g \star h)$
- $f \star 0 = 0$

Remark: There are properties of ordinary multiplication that convolution does not have:

- In general $f \star 1 \neq f$
- $f \star f$ is not necessarily nonnegative.

The Convolution Theorem

Let us again consider the initial value problem

$$y'' + y = g(t), \quad y(0) = 0, \quad y'(0) = 0,$$

Using Laplace transform, we obtain that

$$Y(s) = \frac{1}{1+s^2} G(s), \qquad G(s) = \mathcal{L}\{g(t)\}$$

Therefore the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{1+s^2} G(s)\right\}$$

But we already know that, $y(t) = \int_0^t \sin(t-\tau) g(\tau) d\tau$ is the solution. Thus,

$$\mathcal{L}\left\{\int_0^t \sin(t-\tau) g(\tau) d\tau\right\} = \frac{1}{1+s^2} G(s) = \mathcal{L}\{\sin t\}\mathcal{L}\{g(t)\}$$

Equivalently,

$$\boxed{\mathcal{L}\{\sin t \star g(t)\} = \mathcal{L}\{\sin t\}\mathcal{L}\{g(t)\}}$$

The Convolution Theorem

The Convolution Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ *and* $G(s) = \mathcal{L}\{g(t)\}$ *both exist for* $s > a \geq 0$*, then*

$$\boxed{\mathcal{L}\{f(t) \star g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}}$$

Remark: The convolution theorem can sometimes be effectively used to compute the inverse Laplace transform.

Example: Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s^2+1)^2}$$

Answer:

$$\mathcal{L}^{-1}\{F(s)\} = \int_0^t \sin(t-\tau)\sin\tau d\tau = \ldots$$

Input-Output Problems

Consider the following IVP:

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, \qquad y'(0) = y_1 \tag{2}$$

- Coefficients a, b, and c describe the properties of some physical system
- $g(t)$ is the input to the system
- Values y_0 and y_1 describe the initial state of the system.

In this context, the initial value problem is often referred to as an **input-output problem**. The solution or output $y(t)$ can be separated in two parts: the **free response** and the **forced response**:

$$\boxed{y(t) = \underbrace{\mathcal{L}^{-1}\{H(s)[(as+b)y_0 + ay_1]\}}_{\text{free response}} + \underbrace{\int_0^t h(t-\tau)g(\tau)d\tau}_{\text{forced response}}} \tag{3}$$

where

$$H(s) = \frac{1}{as^2 + bs + c} \qquad \text{is called the transfer function}$$

Input-Output Problems

- Input-output problem: $ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$
- Response: $y(t) = \underbrace{\mathcal{L}^{-1}\{H(s)[(as+b)y_0 + ay_1]\}}_{\text{free response}} + \underbrace{\int_0^t h(t-\tau)g(\tau)d\tau}_{\text{forced response}}$
- $H(s) = \frac{1}{as^2+bs+c}$ is the transfer function, and $h(t) = \mathcal{L}^{-1}\{H(s)\}$

Important observations:

- The free response is the solution of the IVP

$$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_1$$

  - Therefore, $\mathcal{L}^{-1}\{H(s)[(as+b)y_0 + ay_1]\} = c_1 y_1(t) + c_2 y_2(t)$ where $y_1(t)$ and $y_2(t)$ is a fundamental system for the homogeneous ODE

- The forced response is the solution of the IVP

$$ay'' + by' + cy = g(t), \quad y(0) = 0, \quad y'(0) = 0$$

- The transfer function contains all information about the system (a, b, c).

Input-Output Problems

In many applications, the dominant component of the total response is the forced response and the free response is of little importance.

If $g(t) = \delta(t)$, then the forced response is $h(t) = \mathcal{L}^{-1}\{H(s)\}$, and it is the solution of the IVP

$$ay'' + by' + cy = \delta(t), \qquad y(0) = 0, \qquad y'(0) = 0$$

Thus, $h(t)$ is the response of the system to a unit impulse at time $t = 0$ under zero initial conditions. It is natural to call $h(t)$ the **impulse response** of the system.

Forced response y_g is the convolution of the impulse response h and the input g

$$\boxed{y_g(t) = \int_0^t h(t-\tau)g(\tau)d\tau} \quad \text{or in the } s\text{-domain} \quad \boxed{Y_g(s) = H(s)G(s)}$$

Q: How to find the forced response $y_g(t)$?

1. Find the transfer function $H(s)$
2. Find the Laplace transform of the input $G(s)$
3. Then $y_g(t) = \mathcal{L}^{-1}\{H(s)G(s)\}$

Example

Consider the input-output system

$$y'' + 2y' + 5y = t, \quad y(0) = 1, \quad y'(0) = -3$$

- Find the transfer function and the impulse response

  <u>Answer:</u>

$$H(s) = \frac{1}{s^2 + 2s + 5}, \quad h(t) = \frac{1}{2} e^{-t} \sin 2t$$

- Find the forced response

  <u>Answer:</u>

$$y_g(t) = \frac{1}{5} t - \frac{2}{25} + \frac{2}{25} e^{-t} \cos 2t - \frac{3}{50} e^{-t} \sin 2t$$

Summary

- The convolution of f and g is

$$\boxed{(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau}$$

  - $f \star g = g \star f$
  - $f \star (g_1 + g_2) = f \star g_1 + f \star g_2$
  - $(f \star g) \star h = f \star (g \star h)$
  - $f \star 0 = 0$

- Convolution Theorem:

$$\boxed{\mathcal{L}\{f(t) \star g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}}$$

Summary

- Input-output problem:

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, \qquad y'(0) = y_1$$

  - Total response: $y(t) = \underbrace{\mathcal{L}^{-1}\{H(s)[(as+b)y_0 + ay_1]\}}_{\text{free response}} + \underbrace{\int_0^t h(t-\tau)g(\tau)d\tau}_{\text{forced response}}$
  - Transfer function: $H(s) = \frac{1}{as^2+bs+c}$
  - Impulse response: $h(t) = \mathcal{L}^{-1}\{H(s)\}$
    - ⋆ is the solution of

$$ay'' + by' + cy = \delta(t), \qquad y(0) = 0, \qquad y'(0) = 0$$

  - Forced response y_g is the convolution of the impulse response and the input:

$$y_g(t) = \int_0^t h(t-\tau)g(\tau)d\tau \qquad \text{or in the } s\text{−domain} \quad Y_g(s) = H(s)G(s)$$

Homework

Homework:

- Section 5.8
  - 5, 9, 19