Math 245 - Mathematics of Physics and Engineering I

Lecture 28. Impulse Functions

March 28, 2012

Impulsive Functions

In applications, it is necessary to deal with phenomena of an impulsive nature: for example, forces of large magnitude that act over very short period of time. Such problems often lead to ODEs of the form

$$ay'' + by' + cy = g(t)$$

where $g(t)$ is large when $t \in [t_0, t_0 + \varepsilon]$ and is zero otherwise

The integral

$$I(\varepsilon) = \int_{-\infty}^{+\infty} g(t)dt = \int_{t_0}^{t_0+\varepsilon} g(t)dt$$

is a measure of the strength of the forcing function $g(t)$.

- In applications where $g(t)$ represents a force, $I(\varepsilon)$ is referred to as the total impulse of the force over the time interval $[t_0, t_0 + \varepsilon]$ and has units of momentum (force$\times$time).

Modeling Impulsive Functions

Q: How to model a forcing function of large magnitude and which is active only during the short time interval $[t_0, t_0 + \varepsilon]$?

Idea 1: The dominant contribution to the system response $y(t)$ for times $t \geq t_0 + \varepsilon$ is primarily determined by the magnitude of the total impulse $I(\varepsilon)$ rather than the detailed behavior of the forcing function $g(t)$.

Thus, it is natural to model $g(t)$ as follows:

$$g(t) = I_0 \delta_\varepsilon(t - t_0), \qquad \delta_\varepsilon(t) = \frac{u_0(t) - u_\varepsilon(t)}{\varepsilon} = \begin{cases} \frac{1}{\varepsilon}, & 0 \leq t \leq \varepsilon \\ 0, & \text{otherwise.} \end{cases} \qquad (1)$$

- Solve the IVP with $g(t)$ given by (1): find $y_\varepsilon(t)$
- Compute the limiting behavior as $\varepsilon \to 0$: find $y_0(t) = \lim_{\varepsilon \to 0} y_\varepsilon(t)$

Difficulty: It is often tedious.

Idea 2: It turns out that computations get simpler if we change the order:

- first, take the limit $\lim_{\varepsilon \to 0} \delta_\varepsilon(t)$
- second, solve the IVP with corresponding $g(t)$

Unit Impulse Function (Dirac Delta Function)

Let introduce an "idealized object", the **unit impulse function**, that imparts an impulse of magnitude one at t_0, and is zero for $t \neq t_0$.

$$\delta(t - t_0) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t - t_0) \text{``}=\text{''} \begin{cases} +\infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases} \tag{2}$$

Remarks:

- δ is often called the Dirac delta function
- δ is not a function in the ordinary sense. It is an example of what in mathematics are known as generalized functions.
- It is convenient to work with $\delta(t - t_0)$ as with an ordinary function, but it is important to realize that the ultimate justification of such procedures must rest on careful analysis of the limiting operations involved. Such a rigorous mathematical theory exists, but we don't discuss it here.

Properties of $\delta(t - t_0)$

- For any continuous function on an interval $a \le t_0 \le b$,

$$\boxed{\int_a^b f(t)\delta(t-t_0)dt = f(t_0)}$$

- The Laplace transform:

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)dt = e^{-st_0}}$$

- The delta function is the derivative of the unit step function:

$$\boxed{\delta(t-t_0) = u'(t-t_0)}$$

Examples

- Find the solution of the IVP:

$$2y'' + y' + 2y = \delta(t-5), \qquad y(0) = 0, \qquad y'(0) = 0$$

Answer:

$$y(t) = \frac{2}{\sqrt{15}} u_5(t) e^{-(t-5)/4} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)$$

- Find the solution of the IVP:

$$y'' + y = \delta(t-2\pi)\cos t, \qquad y(0) = 0, \qquad y'(0) = 1$$

Answer:

$$y(t) = \sin t + u_{2\pi}(t)\sin t$$

Summary

- Unit Impulse Function or Dirac Delta Function:

$$\delta(t-t_0)=\lim_{\varepsilon\to 0}\delta_\varepsilon(t-t_0)\text{``=''}\begin{cases}+\infty, & t=t_0\\ 0, & t\neq t_0\end{cases}$$

- For any continuous function on an interval $a \le t_0 \le b$,

$$\int_a^b f(t)\delta(t-t_0)dt=f(t_0)$$

- The Laplace transform:

$$\mathcal{L}\{\delta(t-t_0)\}=\int_0^\infty e^{-st}\delta(t-t_0)dt=e^{-st_0}$$

- The delta function is the derivative of the unit step function:

$$\delta(t-t_0)=u'(t-t_0)$$

Homework

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- Section 5.7
  - Find the solution of the given IVP: 1, 3, 5