Math 245 - Mathematics of Physics and Engineering I

Lecture 25. The Inverse Laplace Transform

March 19, 2012

Agenda

- Why do we need $\mathcal{L}^{-1}$?
- Definition of the Inverse Laplace Transform
- Elementary Laplace Transforms
- Linearity of $\mathcal{L}^{-1}$
- Rational Functions
- Summary and Homework

Transformation ODE $\rightsquigarrow$ Algebraic Eq. in the $s-$domain

As we know

- $\mathcal{L}$ is linear
- $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$

These properties of the Laplace transform allow us to convert linear ODE with constant coefficients into algebraic equation in the s-domain.

Example 1:

$$y'' + 2y' + 5y = e^{-t}, \quad y(0) = 1, \quad y'(0) = -3$$

leads to

$$(s^2 + 2s + 5)Y(s) - s + 1 = \frac{1}{s+1}$$

1. The above equation is easy to solve for $Y(s)$
2. $Y(s)$ is the Laplace transform of the solution of the original IVP
3. The solution of the IPV is then

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Definition of the Inverse Laplace Transform

The following theorem allows to define the notion of inverse Laplace transform.

Theorem

Suppose that

- $f(t)$ *and* $g(t)$ *are piecewise continuous and of exponential order on* $[0, \infty)$
- $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$

Then $f(t) = g(t)$ *at all points where both* $f(t)$ *and* $g(t)$ *are continuous.*

If $f(t)$ *and* $g(t)$ *are continuous on* $[0, \infty)$*, then* $f(t) = g(t)$ *for all* $t \in [0, \infty)$

If $F(s) = G(s)$ and f and g are piecewise continuous on $[0, \infty)$, then we identify f with g even though they may differ at countably many points.
(but f and g cannot be different over any interval of positive length!)

Definition

If $f(t)$ is piecewise continuous and of exponential order on $[0, \infty)$ and $\mathcal{L}\{f(t)\} = F(s)$, then we call $f(t)$ the **inverse Laplace transform** of $F(s)$, and denote it by

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Elementary Laplace Transforms

Since there is (essentially) a one-to-one correspondence between functions and their Laplace transforms, we can construct a table of frequently used functions: (see Table on page 328 for more functions):

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}f(t)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a} \quad s > a$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2} \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2} \quad s > a$
$t^n e^{at},\ n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$
$e^{at}f(t)$	$F(s-a)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$

Example 2: Determine $\mathcal{L}^{-1}\{F(s)\}$, where $F(s) = \frac{s+1}{s^2+2s+5}$
Answer: $f(t) = e^{-t}\cos 2t$

Linearity of $\mathcal{L}^{-1}$

$\mathcal{L}^{-1} : F(s) \mapsto f(t)$ is a linear transformation.

Theorem

Assume that $f_1 = \mathcal{L}^{-1}\{F_1\}$ and $f_2 = \mathcal{L}^{-1}\{F_2\}$ are piecewise continuous and of exponential order on $[0, \infty)$. Then for any constants c_1 and c_2

$$\mathcal{L}^{-1}\{c_1 F_1 + c_2 F_2\} = c_1 \mathcal{L}^{-1}\{F_1\} + c_2 \mathcal{L}^{-1}\{F_2\} = c_1 f_1 + c_2 f_2$$

Example 3: Find the inverse Laplace transform of

$$F(s) = \frac{2}{(s+2)^4} + \frac{3}{s^2+16}$$

Answer:

$$f(t) = \frac{1}{3} t^3 e^{-2t} + \frac{3}{4} \sin 4t$$

Rational Functions

Most of the Laplace transforms that arise in the study of differential equations are rational functions

$$Y(s) = \frac{P(s)}{Q(s)},$$

In Example 1,

$$Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$$

Partial Fraction Decomposition:

- If $Q(s) = (s-s_1)(s-s_2)\dots(s-s_n)$, where all s_i are distinct, then

$$Y(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \dots + \frac{A_n}{s-s_n} \tag{1}$$

- If any root s_j of $Q(s)$ is of multiplicity k, i.e. $Q(s) = \dots(s-s_j)^k\dots$, then the j^{th} term in (1) must be changed to

$$\frac{A_j}{s-s_j} \rightsquigarrow \frac{A_{j_1}}{s-s_j} + \frac{A_{j_2}}{(s-s_j)^2} + \dots + \frac{A_{j_k}}{(s-s_j)^k} \tag{2}$$

Rational Functions

Partial Fraction Decomposition

- If $Q(s)$ has a pair of complex conjugate roots $\alpha \pm i\beta$, then the factorization of $Q(s)$ contains factor $(s-\alpha)^2+\beta^2$. If roots $\alpha \pm i\beta$ have multiplicity k, then the partial fraction expansion of $Y(s)$ must include the term

$$\frac{A_1 s + B_1}{(s-\alpha)^2+\beta^2} + \frac{A_2 s + B_2}{[(s-\alpha)^2+\beta^2]^2} + \ldots + \frac{A_k s + B_k}{[(s-\alpha)^2+\beta^2]^k}$$

Example 4: Find the inverse Laplace transform of

$$Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$$

Answer:

$$f(t) = \frac{1}{4}e^{-t} + \frac{3}{4}e^{-t}\cos 2t - e^{-t}\sin 2t$$

Remark: The above function is the solution of the IVP posed in Example 1.

Summary

- If $f(t)$ is piecewise continuous and of exponential order on $[0, \infty)$ and $\mathcal{L}\{f(t)\} = F(s)$, then we call $f(t)$ the **inverse Laplace transform** of $F(s)$:

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

- The above definition makes sense: f is uniquely determined by F
  - "Uniquely": We identify two functions if they differ only at countably many points
- $\mathcal{L}^{-1} : F(s) \mapsto f(t)$ is a linear transformation
- To find $\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\}$, use Partial Fraction Decomposition and "Table Laplace Transforms"

<u>Homework:</u>

- Section 5.3
  - 9, 17, 23