Math 245 - Mathematics of Physics and Engineering I

Lecture 24. Properties of the Laplace Transform

March 9, 2012

Properties of the Laplace Transform

In this Lecture, we will prove several properties of the Laplace transform

$$\boxed{\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)dt}$$

that greatly simplify computation of $\mathcal{L}\{f\}(s)$.

- Laplace transform of $e^{ct}f(t)$
  If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$, and c is a constant, then

  $$\boxed{\mathcal{L}\{e^{ct}f(t)\} = F(s-c)} \quad s > a + c$$

  Geometrically: multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c.

  Example: Find the Laplace transform of $f(t) = e^{-2t}\sin 4t$
  Answer: $F(s) = \frac{4}{(s+2)^2+16}, \quad s > -2.$

Properties of the Laplace Transform

- Laplace transform of $f'(t)$

  Suppose that
  - f is continuous.
  - f' is piecewise continuous on the interval $0 \leq t \leq T$, for any T.
  - f and f' are of exponential order: $|f(t)|, |f'(t)| \leq Me^{at}$.

  Then

  $$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)} \quad s > a$$

  Suppose now that f' is continuous, f'' is piecewise continuous, and both are of exponential order, then

  $$\boxed{\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)} \quad s > a$$

  This can be generalized. Suppose that
  - $f, f', \ldots, f^{(n-1)}$ are continuous
  - $f^{(n)}$ is piecewise continuous on the interval $0 \leq t \leq T$, for any T.
  - $f, f', \ldots, f^{(n)}$ are of exponential order: $|f^{(i)}(t)| \leq Me^{at}$.

  Then

  $$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)} \quad s > a$$

Properties of the Laplace Transform

Example: Consider the following initial value problem:

$$y'' + 2y' + 5y = e^{-t}, \;\; y(0) = 1, \;\; y'(0) = -3$$

Assume that the solution of this problem satisfies the above hypotheses. Find its Laplace transform.

Answer:

$$Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$$

Properties of the Laplace Transform

- Laplace transform of $t^n f(t)$
  Suppose that
  - f is piecewise continuous on the interval $0 \leq t \leq T$
  - f is of exponential order: $|f(t)| \leq Me^{at}$

  Then for any positive integer n

  $$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)} \quad s > a$$

  A simple corollary: for any positive integer n,

  $$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}} \quad s > 0$$

Summary

- If $F(s)=\mathcal{L}\{f(t)\}$ exists for $s>a$, and c is a constant, then

$$\boxed{\mathcal{L}\{e^{ct}f(t)\}=F(s-c)} \quad s>a+c$$

- If
  - $f, f', \ldots, f^{(n-1)}$ are continuous
  - $f^{(n)}$ is piecewise continuous on the interval $0\leq t\leq T$, for any T.
  - $f, f', \ldots, f^{(n)}$ are of exponential order: $|f^{(i)}(t)|\leq Me^{at}$.

  Then

$$\boxed{\mathcal{L}\{f^{(n)}(t)\}=s^n\mathcal{L}\{f(t)\}-s^{n-1}f(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)} \quad s>a$$

- If
  - f is piecewise continuous on the interval $0\leq t\leq T$
  - f is of exponential order: $|f(t)|\leq Me^{at}$

  Then for any positive integer n

$$\boxed{\mathcal{L}\{t^n f(t)\}=(-1)^n F^{(n)}(s)} \quad s>a$$

- For any positive integer n,

$$\boxed{\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}} \quad s>0$$

Homework

Homework:

- Section 5.2
  - 3, 5, 7, 9, 19