Math 245 - Mathematics of Physics and Engineering I

Lecture 23. The Laplace Transform

March 7, 2012

Agenda

- Integral Transforms
- Laplace Transform
- Linearity of the Laplace Transform
- Functions of Exponential Order
- Existence of the Laplace Transform
- Basic Examples
- Summary and Homework

Integral Transforms

An integral transform is a map of the form

$$F(s) = \int_{\alpha}^{\beta} K(t,s) f(t) dt \qquad (1)$$

that maps a given function $f(t)$ into another function $F(s)$.

Q: Why do we call it "transform" and not "operator" as in Lecture 16?

A: Operator: $f(t) \mapsto F(t)$; Transform: $f(t) \mapsto F(s)$.

Terminology:

- $f(t)$ is a function or signal in the time or "t-domain"
- $F(s)$ is its representation in the frequency or "s-domain"
- $K(t,s)$ is called kernel

The Laplace transform is a special case of (1) with

- $\alpha = 0$
- $\beta = \infty$
- $K(t,s) = e^{-st}$

Laplace Transform

Definition

Let f be a function on $[0, \infty)$. The Laplace transform of f is the function F defined by the integral

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt} \tag{2}$$

The domain of $F(s)$ is the set of all values of s for which the integral converges.

Q: Why do wee need the Laplace transform?

A: The Laplace transform is often used in engineering to study input-output relations of linear systems, feedback control systems, and electric circuits.

The Laplace transform is also used for solving ODEs:

1. Transform a "difficult" problem for f (ODE) to a "simpler" (algebraic) problem for F
2. Solve this simpler problem to find F
3. Recover the desired function f from F

Examples

Compute the Laplace transform of

- $f(t) = 1$

$$F(s) = \frac{1}{s} \quad \text{for } s > 0$$

- $f(t) = e^{(a+ib)t}$

$$F(s) = \frac{1}{s - a - ib} \quad \text{for } s > a$$

Linearity of the Laplace Transform

Theorem

Suppose that f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then for any constants c_1 and c_2

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}, \quad s > \max\{a_1, a_2\} \tag{3}$$

Example: Find the Laplace transform of $f(t) = \sin at$, $t \geq 0$.

Hint: Use Euler's formula

$$e^{iat} = \cos at + i \sin at$$

Answer:

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}, \qquad s > 0$$

Functions of Exponential Order

Goal: We want to describe a fairly general class of functions for which the Laplace transform is guaranteed to exist.

Simple observation: $f(t)$ can not grow too fast, since $f(t)e^{-st}$ must vanish sufficiently rapidly as $t \to \infty$ to insure that $\int_0^\infty e^{-st} f(t) dt$ converges.

Definition

A function $f(t)$ is of exponential order (as $t \to \infty$) if

$$|f(t)| \leq Me^{at}, \quad \text{for } t \geq t_0 \tag{4}$$

for some constants t_0, M, and a.

Remark: To show that $f(t)$ is of exponential order, it is suffices to show that $|f(t)|/e^{at}$ is bounded for all sufficiently large t.

Examples: Are the following function of exponential order?

- $f(t) = \cos at$ yes
- $f(t) = t^{10}$ yes
- $f(t) = e^{t^2}$ no

Existence of the Laplace Transform

The following theorem guarantees that the Laplace transform $\mathcal{L}\{f\}$ exists if $f(t)$ is a piecewise continuous function of exponential order.

Theorem

Suppose that

- *f is piecewise continuous on the interval $0 \leq t \leq T$ for any $T > 0$*
- *f is of exponential order, $|f(t)| \leq Me^{at}$ when $t \geq t_0$*

Then the Laplace transform $\mathcal{L}\{f(t)\}$ exists for $s > a$ $\quad (\lim_{s\to\infty} F(s) = 0)$

Reminder: f is piecewise continuous on $[\alpha, \beta]$ if it is continuous at all but possible finitely many points of $[\alpha, \beta]$, at which the function has a finite jump discontinuity

Example: Find the Laplace transform of

$$f(t) = \begin{cases} 4, & 0 \leq t < 1 \\ e^{2t}, & t \geq 1 \end{cases}$$

Summary

- Laplace transform: $f(t) \mapsto F(s)$

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt}$$

  - $f(t)$ is a signal in the t-domain
  - $F(s)$ is its representation in the s-domain

- Laplace transform is linear:

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$$

- $f(t)$ is of exponential order (as $t \to \infty$) if for some constants t_0, M, and a

$$|f(t)| \leq Me^{at}, \quad \text{for } t \geq t_0$$

- Laplace transform $\mathcal{L}\{f\}$ exists if $f(t)$ is a piecewise continuous function of exponential order.

Homework

Homework:

- Section 5.1
  - 7, 10, 17, 25