Math 245 - Mathematics of Physics and Engineering I

Lecture 22. Variation of Parameters for Linear Second Order Equations

March 5, 2012

Variation of Parameters for Systems

In Lecture 21, we learned how to find a particular solution of a nonhomogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

If

- each entry of $\mathbf{P}(t)$ and $\mathbf{g}(t)$ be a continuous function on an interval I
- $\mathbf{x}_1$ and $\mathbf{x}_2$ be a fundamental set of solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$
- $\mathbf{X}(t) = (\mathbf{x}_1, \mathbf{x}_2) = \begin{pmatrix} x_1^1(t) & x_2^1(t) \\ x_1^2(t) & x_2^2(t) \end{pmatrix}$ be the corresponding fundamental matrix

Then

- A particular solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is

$$\boxed{\mathbf{x}_p(t) = \mathbf{X}(t) \int \mathbf{X}^{-1}(t)\mathbf{g}(t)dt}$$

- The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is

$$\boxed{\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{x}_p(t)}$$

Variation of Parameters for Equations

Theorem

Consider the following nonhomogeneous equation:

$$y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

If

- $p(t)$, $q(t)$, *and* $g(t)$ *are continuous on an interval* I
- $y_1(t)$ *and* $y_2(t)$ *are a fundamental set of solutions of the homogeneous equation* $y'' + p(t)y' + q(t)y = 0$

Then

- *a particular solution of (1) is*

$$\boxed{Y(t) = y_2(t)\int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}dt - y_1(t)\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}dt} \tag{2}$$

- *the general solution is*

$$\boxed{y = c_1 y_1(t) + c_2 y_2(t) + Y(t)} \tag{3}$$

Examples

- Find the general solution:

$$y'' - 2y' + y = \frac{e^t}{1+t^2}$$

Answer:

$$y(t) = c_1 e^t + c_2 t e^t + t e^t \arctan t - \frac{e^t \ln(1+t^2)}{2}$$

- Consider the equation

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 3x^{3/2} \sin x, \qquad x > 0$$

  - Verify that

$$y_1(x) = x^{-1/2} \sin x \qquad \text{and} \qquad y_2(x) = x^{-1/2} \cos x$$

    satisfy the corresponding homogeneous equation
  - Find a particular solution of the nonhomogeneous equation

    Answer:

$$Y(t) = -\frac{3}{2} x^{1/2} \cos x$$

Summary: Variation of Parameters for Equations

How to find a particular solution of

$$y'' + p(t)y' + q(t)y = g(t)$$

1. Find a fundamental set of solution $y_1(t)$ and $y_2(t)$ of the corresponding homogeneous equation
   - if $p(t)$ and $q(t)$ are constants, then it is easy (see Lecture 18)
   - if not, then, in general, it is difficult
2. A particular solution is then

$$\boxed{Y(t) = y_2(t)\int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}dt - y_1(t)\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}dt}$$

   where W is the Wronskian

$$W[y_1,y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Homework

Homework:

- Section 4.7
  - 11, 16, 23