Math 245 - Mathematics of Physics and Engineering I

Lecture 21. Variation of Parameters for Linear First Order Systems

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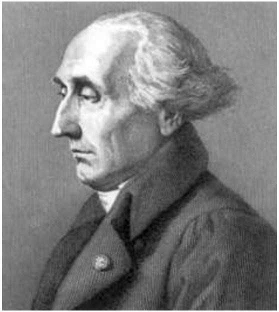
Variation of Parameters

In this Lecture, we will learn another method for finding a particular solution of a nonhomogeneous equation, known as **variation of parameters** or **variation of constants**. This method is due to Lagrange



- The main advantage of this method is that it is a general method. In principle, at least, it can be applied to any nonhomogeneous equation.
- The main drawback is that computations are often tedious.

Variation of Parameters for Systems

Let us first consider the nonhomogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t) \tag{1}$$

where

$$\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \quad \text{and} \quad g(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \tag{2}$$

Suppose that

$$\mathbf{x}_1(t) = \begin{pmatrix} x_1^1(t) \\ x_1^2(t) \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} x_2^1(t) \\ x_2^2(t) \end{pmatrix} \tag{3}$$

form a fundamental set of solutions for the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. This means that

$$\mathbf{x}_1' = \mathbf{P}(t)\mathbf{x}_1, \quad \mathbf{x}_2' = \mathbf{P}(t)\mathbf{x}_2, \quad W[\mathbf{x}_1, \mathbf{x}_2] = \det \mathbf{X}(t) = \begin{vmatrix} x_1^1(t) & x_2^1(t) \\ x_1^2(t) & x_2^2(t) \end{vmatrix} \neq 0 \tag{4}$$

- Matrix $\mathbf{X}(t) = \begin{pmatrix} x_1^1(t) & x_2^1(t) \\ x_1^2(t) & x_2^2(t) \end{pmatrix}$ is called a fundamental matrix

The Method

In terms of a fundamental matrix $\mathbf{X}(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t))$, the fact that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t))$ are a fundamental set of solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ is written as follows:

$$\mathbf{X}'(t) = \mathbf{P}(t)\mathbf{X}(t), \qquad \det \mathbf{X}(t) \neq 0 \tag{5}$$

The method of variation of parameters consists of 3 steps:

1. Find a fundamental set of solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.
   Then the general solution of the homogeneous equation is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \tag{6}$$

2. To find a particular solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, replace the c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$. In other words, vary the parameters c_1 and c_2:

$$\mathbf{x}_p(t) = u_1(t)\mathbf{x}_1(t) + u_2(t)\mathbf{x}_2(t) = \mathbf{X}(t)\mathbf{u}(t) \tag{7}$$

3. Find functions $u_1(t)$ and $u_2(t)$ such that (7) is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$

Main Result

Theorem

If each entry of $\mathbf{P}(t)$ and $\mathbf{g}(t)$ is a continuous function on an interval I, then

- *A particular solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is*

$$\boxed{\mathbf{x}_p(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{g}(t)dt} \qquad \mathbf{X}(t) = \begin{pmatrix} x_1^1(t) & x_2^1(t) \\ x_1^2(t) & x_2^2(t) \end{pmatrix} \tag{8}$$

- *The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is*

$$\boxed{\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{x}_p(t)} \tag{9}$$

Remark: There are two major problems with using this method:

- If $\mathbf{P}(t)$ is not a constant matrix, then it is difficult to find a fundamental set of solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.
- The evaluation of the integrals appearing in (5) may be difficult.

Example

Find the solution of the initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 2 & -5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10\cos t \\ 2e^{-t} \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

Summary: Variation of Parameters for Systems

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

Let

- each entry of $\mathbf{P}(t)$ and $\mathbf{g}(t)$ be a continuous function on an interval I
- $\mathbf{x}_1$ and $\mathbf{x}_2$ be a fundamental set of solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$
- $\mathbf{X}(t) = (\mathbf{x}_1, \mathbf{x}_2) = \begin{pmatrix} x_1^1(t) & x_2^1(t) \\ x_1^2(t) & x_2^2(t) \end{pmatrix}$ be the corresponding fundamental matrix

Then

- A particular solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is

$$\boxed{\mathbf{x}_p(t) = \mathbf{X}(t) \int \mathbf{X}^{-1}(t)\mathbf{g}(t)dt}$$

- The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is

$$\boxed{\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{x}_p(t)}$$

Homework

Homework:

- Section 4.7
  - 3, 5, 9