Math 245 - Mathematics of Physics and Engineering I

Lecture 20. Method of Undetermined Coefficients-II

February 29, 2012

The Method of Undetermined Coefficients is used to find a particular solution of a nonhomogeneous equation

$$ay'' + by' + cy = g(t)$$

In Lecture 19, we obtained the following Half-Way Results:

- If $g(t) = e^{\alpha t}$, then assume that $Y(t) = Ae^{\alpha t}$
- If $g(t) = \sin \beta t$ or $g(t) = \cos \beta t$, then assume that $Y(t) = A \sin \beta t + B \cos \beta t$
- If $g(t)$ is a polynomial, then assume that $Y(t)$ is a polynomial of the same degree.
- If $g(t)$ is a product of the above functions, $g(t) = g_1(t)g_2(t)$, then assume that $Y(t)$ is the corresponding product $Y(t) = Y_1(t)Y_2(t)$.

However there is one difficulty that sometimes occurs.
The following example illustrates how it arises.

Example: Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t}$$

The above guidelines do not work in this example because the assumed solution Ae^{-t} is actually a solution of the corresponding homogeneous equation!

The above example suggests that we need to modify the guidelines:

- if the assumed particular solution duplicates a solution of the corresponding homogeneous equation, then multiply the particular solution by t.
- sometimes, this modification will be insufficient, in which case it is necessary to multiply by t a second time.

The particular solution of $ay'' + by' + cy = g(t)$

	$g(t)$	$Y(t)$
1	$P_n(t)$	$t^s G_n(t)$
2	$P_n(t)e^{\alpha t}$	$t^s G_n(t)e^{\alpha t}$
3	$P_n(t)e^{\alpha t}\sin\beta t$	$t^s\left[G_n(t)e^{\alpha t}\cos\beta t + H_n(t)e^{\alpha t}\sin\beta t\right]$
4	$P_n(t)e^{\alpha t}\cos\beta t$	$t^s\left[G_n(t)e^{\alpha t}\cos\beta t + H_n(t)e^{\alpha t}\sin\beta t\right]$

- $P_n(t), G_n(t), H_n(t)$ are polynomials of degree n
- $s = 0, 1, 2$ is the smallest integer that will ensure that no term in $Y(t)$ is a solution of the corresponding homogeneous equation:
  - Case 1: $s = \#$ times 0 is a root of the characteristic equation
  - Case 2: $s = \#$ times α is a root of the characteristic equation
  - Cases 3,4: $s = \#$ times $\alpha + i\beta$ is a root of the characteristic equation

Examples

Find a suitable form for the particular solution

- $y'' = 3t^3 - t$
  - Answer:

$$Y = A_0 t^5 + A_1 t^4 + A_2 t^3 + A_3 t^2$$

- $y'' + 2y' + 5y = t^2 e^{-t} \sin 2t$
  - Answer:

$$Y = t\left[(A_0 t^2 + A_1 t + A_2)e^{-t}\cos 2t + (B_0 t^2 + B_1 t + B_2)e^{-t}\sin 2t\right]$$

- $y'' + y = \tan t$
  - The method of undetermined coefficients is not applicable; but a particular solution can be found by the method of variation of the parameters

Superposition Principle for Nonhomogeneous Equations

Suppose that $g(t)$ is the sum of two terms, $g(t) = g_1(t) + g_2(t)$, and suppose that $Y_1(t)$ is a solution of

$$ay'' + by' + cy = g_1(t)$$

and $Y_2(t)$ is a solution of

$$ay'' + by' + cy = g_2(t)$$

Then $Y(t) = Y_1(t) + Y_2(t)$ is a solution of

$$ay'' + by' + cy = g_1(t) + g_2(t)$$

This property is called the superposition principle for nonhomogeneous equations.

Practical Significance:

For an equation whose function $g(t)$ can be expressed as a sum, we can consider instead several simpler equations and then add together the results.

Example

Find a suitable form for the particular solution of the following equation

$$y'' - y' - 2y = -3te^{-t} + 2\cos 4t$$

Answer:

$$Y(t) = t(A_0 t + A_1)e^{-t} + B_0 \cos 4t + B_1 \sin 4t$$

Summary: Method of Undetermined Coefficients

To find a particular solution of a nonhomogeneous equation

$$ay'' + by' + cy = g(t)$$

do the following:

1. Make sure that $g(t)$ involves nothing more than exponential functions $e^{\alpha t}$, sines $\sin \beta t$, cosines $\cos \beta t$, polynomials $P_n(t) = a_0 t^n + a_1 t^{n-1} + \ldots + a_n$, or sums or products of such functions. If this is not the case, use the method of variation of parameters (Lectures 21,22).
2. If $g(t) = g_1(t) + g_2(t) + \ldots + g_n(t)$, then the original problem beaks down to n subproblems: the i^{th} subproblem is to find a particular solution $Y_i(t)$ of

$$ay'' + by' + cy = g_i(t)$$

3. Find $Y_i(t)$ using the table on slide 3
4. $Y(t) = Y_1(t) + \ldots + Y_n(t)$ is a particular solution of the original nonhomogeneous equation.

Homework

Homework:

- Section 4.5
  - 3, 5, 15, 23(a)