Math 245 - Mathematics of Physics and Engineering I

Lecture 19. Nonhomogeneous Equations: Method of Undetermined Coefficients

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Nonhomogeneous Equations

In Lecture 18, we learned how to solve linear homogeneous second order ODE with constant coefficients,

$$ay'' + by' + cy = 0 \tag{1}$$

In this lecture, our goal is learn how to solve nonhomogeneous equations:

$$ay'' + by' + cy = g(t) \tag{2}$$

The structure of the general solution of (2) is described by the following theorem:

Theorem

- *If Y_1 and Y_2 are two solutions of (2), then $(Y_1 - Y_2)$ is a solution of (1).*
- *The general solution of (2) can be written in the form*

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)} \tag{3}$$

  - *$y_1(t)$ and $y_2(t)$ form a fundamental set of (1)*
  - *$Y(t)$ is some specific solution of (2)*
  - *c_1 and c_2 are arbitrary constants*

Nonhomogeneous Equations

General Strategy for Solving $ay'' + by' + cy = g(t)$:

1. Find the general solution $c_1y_1 + c_2y_2$ of the corresponding homogeneous equation $ay'' + by' + cy = 0$. This solution is called the complementary solution.
2. Find some single solution Y of the nonhomogeneous equation. Often this solution is referred to as a particular solution.
3. The general solution of $ay'' + by' + cy = g(t)$ is then $y = c_1y_1 + c_2y_2 + Y$.

Question: How to find a particular solution Y?

We will discuss two methods:

- Method of Undetermined Coefficients
  - Advantage: easy to use
  - Disadvantage: sometimes does not work
- Method of Variation of Parameters
  - Advantage: general method (always works)
  - Disadvantage: computationally difficult

Method of Undetermined Coefficients: Main Idea

In the method of undetermined coefficients, we assume that the particular solution Y has a specific form, but with coefficient left unspecified,

$$Y(t) = f(t; A, B, C, \ldots) \tag{4}$$

- f is some function that depends on parameters $A, B, C, \ldots$

We then substitute the assumed expression (4) into our equation

$$ay'' + by' + cy = g(t)$$

and attempt to determine the coefficients $A, B, C, \ldots$ so as to satisfy that equation. There are two possible outcomes:

- If we are successful $\Rightarrow$ we have found Y
- If not $\Rightarrow$ there is no solution of the form (4). In this case, we may modify (4) and try again.

Important Remark:

This method is usually used only for equation in which $g(t)$ consists of polynomials, exponential functions, sines, cosines, or sums or products of these functions.

Examples

- Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t}$$

- Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t$$

- Find a particular solution of

$$y'' - 3y' - 4y = 4t^2 - 1$$

Method of Undetermined Coefficients

$$ay'' + by' + cy = g(t)$$

Half-Way Results:

- If $g(t) = e^{\alpha t}$, then assume that $Y(t) = Ae^{\alpha t}$
- If $g(t) = \sin\beta t$ or $g(t) = \cos\beta t$, then assume that $Y(t) = A\sin\beta t + B\cos\beta t$
- If $g(t)$ is a polynomial, then assume that $Y(t)$ is a polynomial of the same degree.

Important Remark: As we will see next time, these guidelines will not lead us to success every time: in some cases, we will not be able to find a particular solution. In such cases, we will need to slightly modify the expressions for $Y(t)$.

Question: What if $g(t)$ is a product of $e^{\alpha t}$, $\sin\beta t$, $\cos\beta t$, and polynomials?

Example: Find a particular solution of

$$y'' - 3y' - 4y = -8e^t\cos 2t$$

Summary

- The general solution of

$$ay'' + by' + cy = g(t)$$

can be written in the form

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)} \qquad (5)$$

  - $y_1(t)$ and $y_2(t)$ form a fundamental set of the corresponding homogeneous equation $ay'' + by' + cy = 0$
  - $Y(t)$ is some specific solution of the nonhomogeneous equation
  - c_1 and c_2 are arbitrary constants
- How to find a particular solution $Y(t)$?
  - Method of Undetermined Coefficients
    Half-Way Results:
    - If $g(t) = e^{\alpha t}$, then assume that $Y(t) = Ae^{\alpha t}$
    - If $g(t) = \sin\beta t$ or $g(t) = \cos\beta t$, then assume that $Y(t) = A\sin\beta t + B\cos\beta t$
    - If $g(t)$ is a polynomial, then assume that $Y(t)$ is a polynomial of the same degree.

Homework

<u>Homework:</u>

- Section 4.5
  - 1, 2, 13