
February 27, 2012
Nonhomogeneous Equations

In Lecture 18, we learned how to solve linear homogeneous second order ODE with constant coefficients,

\[ ay'' + by' + cy = 0 \quad (1) \]

In this lecture, our goal is learn how to solve nonhomogeneous equations:

\[ ay'' + by' + cy = g(t) \quad (2) \]

The structure of the general solution of (2) is described by the following theorem:

**Theorem**

- If \( Y_1 \) and \( Y_2 \) are two solutions of (2), then \( (Y_1 - Y_2) \) is a solution of (1).
- The general solution of (2) can be written in the form

\[
y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) \quad (3)
\]

- \( y_1(t) \) and \( y_2(t) \) form a fundamental set of (1)
- \( Y(t) \) is some specific solution of (2)
- \( c_1 \) and \( c_2 \) are arbitrary constants
Nonhomogeneous Equations

General Strategy for Solving $ay'' + by' + cy = g(t)$:

1. Find the general solution $c_1y_1 + c_2y_2$ of the corresponding homogeneous equation $ay'' + by' + cy = 0$. This solution is called the complementary solution.

2. Find some single solution $Y$ of the nonhomogeneous equation. Often this solution is referred to as a particular solution.

3. The general solution of $ay'' + by' + cy = g(t)$ is then $y = c_1y_1 + c_2y_2 + Y$.

Question: How to find a particular solution $Y$?

We will discuss two methods:

- **Method of Undetermined Coefficients**
  - Advantage: easy to use
  - Disadvantage: sometimes does not work

- **Method of Variation of Parameters**
  - Advantage: general method (always works)
  - Disadvantage: computationally difficult
Method of Undetermined Coefficients: Main Idea

In the method of undetermined coefficients, we assume that the particular solution \( Y \) has a specific form, but with coefficient left unspecified,

\[
Y(t) = f(t; A, B, C, \ldots)
\]  

\( f \) is some function that depends on parameters \( A, B, C, \ldots \)

We then substitute the assumed expression (4) into our equation

\[
ay'' + by' + cy = g(t)
\]

and attempt to determine the coefficients \( A, B, C, \ldots \) so as to satisfy that equation. There are two possible outcomes:

- If we are successful \( \Rightarrow \) we have found \( Y \)
- If not \( \Rightarrow \) there is no solution of the form (4). In this case, we may modify (4) and try again.

Important Remark:

This method is usually used only for equation in which \( g(t) \) consists of polynomials, exponential functions, sines, cosines, or sums or products of these functions.
Examples

- Find a particular solution of

\[ y'' - 3y' - 4y = 3e^{2t} \]

- Find a particular solution of

\[ y'' - 3y' - 4y = 2 \sin t \]

- Find a particular solution of

\[ y'' - 3y' - 4y = 4t^2 - 1 \]
Method of Undetermined Coefficients

\[ ay'' + by' + cy = g(t) \]

Half-Way Results:

- If \( g(t) = e^{\alpha t} \), then assume that \( Y(t) = Ae^{\alpha t} \)
- If \( g(t) = \sin \beta t \) or \( g(t) = \cos \beta t \), then assume that \( Y(t) = A\sin \beta t + B\cos \beta t \)
- If \( g(t) \) is a polynomial, then assume that \( Y(t) \) is a polynomial of the same degree.

Important Remark: As we will see next time, these guidelines will not lead us to success every time: in some cases, we will not be able to find a particular solution. In such cases, we will need to slightly modify the expressions for \( Y(t) \).

Question: What if \( g(t) \) is a product of \( e^{\alpha t} \), \( \sin \beta t \), \( \cos \beta t \), and polynomials?

Example: Find a particular solution of

\[ y'' - 3y' - 4y = -8e^t \cos 2t \]
Summary

- The general solution of
  \[ ay'' + by' + cy = g(t) \]
  can be written in the form
  \[ y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \] (5)

  - \( y_1(t) \) and \( y_2(t) \) form a fundamental set of the corresponding homogeneous equation \( ay'' + by' + cy = 0 \)
  - \( Y(t) \) is some specific solution of the nonhomogeneous equation
  - \( c_1 \) and \( c_2 \) are arbitrary constants

- How to find a particular solution \( Y(t) ? \)
  - Method of Undetermined Coefficients
  Half-Way Results:
    - If \( g(t) = e^{\alpha t} \), then assume that \( Y(t) = Ae^{\alpha t} \)
    - If \( g(t) = \sin \beta t \) or \( g(t) = \cos \beta t \), then assume that \( Y(t) = A \sin \beta t + B \cos \beta t \)
    - If \( g(t) \) is a polynomial, then assume that \( Y(t) \) is a polynomial of the same degree.
Homework

Homework:

- Section 4.5
  - 1, 2, 13