Math 245 - Mathematics of Physics and Engineering I

Lecture 18. Linear Homogeneous Second Order ODEs with Constant Coefficients

February 24, 2012

In this Lecture, we study the problem of finding a fundamental set of solutions of the linear homogeneous second order ODE with constant coefficients

$$ay'' + by' + cy = 0, \qquad a \neq 0 \tag{1}$$

Using the state variables $x_1 = y$ and $x_2 = y'$, we transform (1) into the first order linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \mathbf{x} \tag{2}$$

As we know (Lecture 16),

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3}$$

is a general solution of (1) if and only if

$$\mathbf{x} = c_1 \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix} + c_2 \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix} \tag{4}$$

is a general solution of (2).

We know how to find general solutions of systems (2): the eigenvalue method.

Theorem

- *The eigenvalues of* $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$ *are the roots of*

$$\boxed{Z(\lambda) = a\lambda^2 + b\lambda + c = 0} \tag{5}$$

- *If* λ *is an eigenvalue of* A*, then*
  - *the corresponding eigenvector is* $v = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$
  - $\mathbf{x} = \begin{pmatrix} e^{\lambda t} \\ \lambda e^{\lambda t} \end{pmatrix}$ *is a solution of* $\mathbf{x}' = \mathbf{A}\mathbf{x}$
  - $\boxed{y = e^{\lambda t}}$ *is a solution of* $ay'' + by' + cy = 0$

Eq. (5) is called the characteristic equation for the ODE $ay'' + by' + cy = 0$
The roots of the characteristic equation (=eigenvalues of $\mathbf{A}$) are:

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{6}$$

Case I: Distinct Real Roots, $b^2 - 4ac > 0$

The eigenvectors corresponding to the eigenvalues λ_1 and λ_2 are

$$v_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is then

$$\mathbf{x} = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

and the general solution of $ay'' + by' + cy = 0$ is

$$\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

Case II: Repeated Roots, $b^2 - 4ac = 0$

When we studied autonomous homogeneous systems with repeated eigenvalues $\lambda_1 = \lambda_2$ (Lecture 13), we considered two cases:

- $\mathbf{A}$ is diagonal $\Leftrightarrow$ λ has two independent eigenvectors
- $\mathbf{A}$ is nondiagonal $\Leftrightarrow$ λ has one independent eigenvector $\mathbf{v}$

In our case, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$ is always nondiagonal.

In this case, the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x} = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}(t\mathbf{v} + \mathbf{w}) \tag{7}$$

where $\mathbf{w}$ is a generalized eigenvector corresponding to λ, that is any solution of

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{w} = \mathbf{v}$$

In our case,

$$\lambda = -b/2a, \quad \mathbf{v} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, the general solution of $ay'' + by' + cy = 0$ is

$$\boxed{y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}}$$

Case III: Complex Conjugate Roots: $b^2 - 4ac < 0$

In this case, the roots of the characteristic equation are

$$\lambda_1 = \underbrace{\frac{-b}{2a}}_{\alpha} + i \underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{\beta} \quad \text{and} \quad \lambda_2 = \underbrace{\frac{-b}{2a}}_{\alpha} - i \underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{\beta}$$

The corresponding eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \alpha + i\beta \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ \alpha \end{pmatrix}}_{\mathbf{a}} + i \underbrace{\begin{pmatrix} 1 \\ \beta \end{pmatrix}}_{\mathbf{b}} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ \alpha - i\beta \end{pmatrix}$$

Then, the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x} = c_1 e^{\alpha t}(\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t) + c_2 e^{\alpha t}(\mathbf{a} \sin \beta t + \mathbf{b} \cos \beta t)$$

Thus, the general solution of $ay'' + by' + cy = 0$ is

$$\boxed{y(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t}$$

Examples

Find the general solutions for the following ODEs:

- $y'' + 5y' + 6y = 0$
- $y'' + y' + y = 0$
- $4y'' - 4y' + y = 0$
  - Find the solution of the IVP, $y(0) = 2$, $y'(0) = 1/3$.

Find an ODE whose general solution is

$$y = c_1 e^{-3t} \cos 4t + c_2 e^{-3t} \sin 4t$$

Find the values of α for which all solutions tend to zero as $t \to \infty$.

$$y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$$

Summary

- The general solution of the ODE

$$ay'' + by' + cy = 0$$

is

  - Distinct Real Roots, $\lambda_1 \neq \lambda_2$, $b^2 - 4ac > 0$

$$\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

  - Repeated Roots, $\lambda_1 = \lambda_2 = \lambda$, $b^2 - 4ac = 0$

$$\boxed{y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}}$$

  - Complex Conjugate Roots, $\lambda = \alpha \pm i\beta$, $b^2 - 4ac > 0$

$$\boxed{y(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t}$$

Homework

<u>Homework:</u>

- Section 4.3
  - 9(a), 15(a), 17(a)
  - Solve the IVP 37
  - 45, 47