Math 245 - Mathematics of Physics and Engineering I

Lecture 17. Theory of Second Order Linear Homogeneous ODEs - II

February 22, 2012

Agenda

- General Solution
- Wronskians
- Fundamental Set of Solutions
- Abel's Theorem
- Summary and Homework

General Solution

We study second order linear homogeneous ODEs

$$y'' + p(t)y' + q(t)y = 0 \qquad (1)$$

In Lecture 16, we discussed the Principle of Superposition:
If y_1 and y_2 are two solutions of (1) then any linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \qquad (2)$$

is also a solution. This principle allows us to enlarge the set of solutions that we have, provided that we are able to find two solutions to start with.

Question:

Does (2) give us the general solutions of (1)? In other words, can any solution of (1) be written in the form of (2) by an appropriate choice of the constants?

Strategy:

1. Obtain the result for the general system of two first order linear homogeneous ODEs
2. The result for (1) will follow from the general case, since (1) can be transformed to such system.

Homogeneous Systems

Consider the homogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{3}$$

where all entities of $\mathbf{P}(t)$ are continuous functions on I. Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two solutions of (3) on I. Then (principle of superposition) for any constants c_1, c_2

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \tag{4}$$

is also a solution. Suppose that $\mathbf{z}(t)$ is a solution of (3) on I. The initial condition

$$\mathbf{z}(t_0) = \mathbf{z}_0, \qquad t_0 \in I \tag{5}$$

determines $\mathbf{z}(t)$ uniquely. Thus, to show that $\mathbf{z}$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$, it is enough to show that we can find c_1 and c_2 such that

$$\mathbf{x}(t_0) = c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) = \mathbf{z}_0 \tag{6}$$

Remark: Why "enough"? See the existence and uniqueness theorem (Lecture 16).

Homogeneous Systems

Writing (6) in detail, we have

$$\begin{pmatrix} x_1^1(t_0) & x_2^1(t_0) \\ x_1^2(t_0) & x_2^2(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} z_0^1 \\ z_0^2 \end{pmatrix} \tag{7}$$

System (7) has a unique solution for c_1 and c_2 if and only if the matrix on the right hand side is nonsingular for any $t_0 \in I$. The determinant

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} x_1^1(t) & x_2^1(t) \\ x_1^2(t) & x_2^2(t) \end{vmatrix} \tag{8}$$

is called the Wronskian of $\mathbf{x}_1$ and $\mathbf{x}_2$. Thus, we proved the following

Theorem

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, where $\mathbf{P}(t)$ has continuous entities on I. If their Wronskian is not zero on I, $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$, then $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions:

- *The general solution is given by $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$*
- *Initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ determines c_1 and c_2 uniquely.*

Second Order Linear Homogeneous ODE

To discuss the general solution of

$$y'' + p(t)y' + q(t)y = 0 \tag{9}$$

let us apply the theorem to the following special homogeneous system:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x} \tag{10}$$

Remark: (9) and (10) are "equivalent":
$y(t)$ is a solution of (9) $\Leftrightarrow$ $\mathbf{x} = (y(t), y'(t))^T$ is a solution of (10).

Q: How does the Wronskian of y_1 and y_2 look like?

A:

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

General Solution for a 2^{nd} Order Linear Homogeneous ODE

Theorem

Suppose that y_1 and y_2 are two solutions of

$$y'' + p(t)y' + q(t)y = 0$$

where $p(t)$ and $q(t)$ are continuous functions on I. If their Wronskian is not zero on I,

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) \neq 0,$$

then y_1 and y_2 form a fundamental set of solutions:

- *The general solution is given by $y = c_1 y_1 + c_2 y_2$*
- *Initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$ determines c_1 and c_2 uniquely.*

A remarkable property of the Wronskian

From these theorems, it is clear that the Wronskian is a key object.
It has one nice property.

Recall that it was defined in terms of solutions:

- for a homogeneous system of first order equations

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} x_1^1(t) & x_2^1(t) \\ x_1^2(t) & x_2^2(t) \end{vmatrix}$$

- for the second order homogeneous equation

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

It turns out, that (without knowing any solutions!)

the Wronskian can be determined directly from the differential equation(s)

Abel's Theorem

Abel's Theorem

- *The Wronskian of two solutions of the homogeneous system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

is given by

$$\boxed{W(t) = c \exp\left(\int \mathrm{tr}(\mathbf{P}(t))dt\right)}$$

where constant c depends on the pair of solutions.

- *The Wronskian of two solutions of the second order equation*

$$y'' + p(t)y' + q(t)y = 0$$

is given by

$$\boxed{W(t) = c \exp\left(-\int p(t)dt\right)}$$

where constant c depends on the pair of solutions.

Example

- Find the Wronskian of any pair of solutions of

$$(1-t)y'' + ty' - y = 0$$

Important Corollary

An important property of the Wronskian follows immediately from the Abel's theorem:

Corollary

- *If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

  and all entities of $\mathbf{P}(t)$ are continuous on I, then the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2]$ is either never zero or always zero in I.

- *If y_1 and y_2 are two solutions of the second order equation*

$$y'' + p(t)y' + q(t)y = 0$$

  where p and q are continuous on I, then the Wronskian $W[y_1, y_2]$ is either never zero or always zero in I.

Concluding Remarks

To find the general solution of $y'' + p(t)y' + q(t)y = 0$, all we need to do is to find two "nice" solutions y_1 and y_2. "Nice" = Wronskian is not zero.

For a system of two first order linear ODEs with constant coefficients, it was relatively easy to find a fundamental set of solutions by using the eigenvalue method (Lectures 10,11,13). However, for systems with variable coefficients, the task of finding a suitable pair of solutions is much more challenging.

Summary

- $\boxed{\mathbf{x}' = \mathbf{P}(t)\mathbf{x}}$
  - $\mathbf{P}(t)$ is continuous on I
  - $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions

  If $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$, then $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions.
- $\boxed{y'' + p(t)y' + q(t)y = 0}$
  - $p(t)$ and $q(t)$ are continuous functions on I.
  - y_1 and y_2 are two solutions

  If $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) \neq 0$, then y_1 and y_2 form a fundamental set of solutions.
- Abel's Theorem:
  - $W(t) = c \exp\left(\int \operatorname{tr}(\mathbf{P}(t))dt\right)$
  - $W(t) = c \exp\left(-\int p(t)dt\right)$
- The Wronskian is either never zero or always zero.

Homework

<u>Homework:</u>

- Section 4.2
  - 11, 15, 17, 23