Math 245 - Mathematics of Physics and Engineering I

Lecture 16. Theory of Second Order Linear Homogeneous ODEs

February 17, 2012

Agenda

- Existence and Uniqueness of Solutions
- Linear Operators
- Principle of Superposition for Homogeneous Equations
- Summary and Homework

Existence and Uniqueness

Consider the second order linear ODE:

$$y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

where p, q and g are continuous functions on the interval I. By introducing the state variables

$$x_1 = y, \qquad x_2 = y' \tag{2}$$

we convert (1) to the system of two first order linear ODEs

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \tag{3}$$

where $\mathbf{x} = (x_1, x_2)^T$. Trivial, yet very important observation:

$y = \phi(t)$ is a solution of (1) if and only if $\mathbf{x} = (\phi(t), \phi'(t))^T$ is a solution of (3)

Existence and Uniqueness

If in addition to equation (1) we also have initial conditions

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_0, \\ y'(t_0) = y_1. \end{cases} \tag{4}$$

then the initial value problem (4) is equivalent to

$$\begin{cases} \mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}, \\ \mathbf{x}(t_0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \end{cases} \tag{5}$$

"Equivalent" in the following sense:

$y = \phi(t)$ is a solution of (4) if and only if $\mathbf{x} = (\phi(t), \phi'(t))^T$ is a solution of (5)

Existence and Uniqueness

Observe that (3) is a special case of the general system of two first order ODEs:

$$\mathbf{x}' = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t) \qquad (6)$$

In Lecture 9, we described sufficient conditions for the existence of a unique solution to an initial value problem for (6):

Theorem

Let

- *all functions* $p_{11}(t), p_{12}(t), p_{21}(t), p_{22}(t), g_1(t)$, *and* $g_2(t)$ *be continuous on an open interval* $I = (\alpha, \beta)$, $t_0 \in I$, *and*
- $\mathbf{x}_0$ *be any given vector*

Then there **exists a unique solution** *of the initial value problem*

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

on the interval I.

Existence and Uniqueness

As a corollary, we obtain the following result:

Corollary

- *Let $p(t), q(t)$, and $g(t)$ be continuous functions on an open interval I*
- *Let t_0 be any point in I, and*
- *Let y_0 and y_1 be any given numbers.*

Then there **exists a unique solution** *of the initial value problem*

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_0, \\ y'(t_0) = y_1. \end{cases}$$

on the interval I.

Example: Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0, \quad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

Linear Operators and Principle of Superposition

Definition

An **operator** A is a map that associates one function to another function,

$$A : f \to g = A[f]$$

Examples:

- Operator of multiplication by function h:

$$h : f \to h[f], \qquad h[f](t) = h(t)f(t)$$

- Operator of differentiation:

$$D : f \to D[f], \qquad D[f](t) = \frac{df}{dt}(t)$$

Definition

Operator A is called **linear** if

$$A[c_1 f_1 + c_2 f_2] = c_1 A[f_1] + c_2 A[f_2]$$

Both $h[\cdot]$ and $D[\cdot]$ are linear operators

Linear Operators and Principle of Superposition

Using $h[\cdot]$ and $D[\cdot]$, we can construct new linear operators. For example, if p and q are continuous functions on an interval I, we can define the second order differential operator:

$$L = D^2 + pD + q \equiv \frac{d^2}{dt^2} + p\frac{d}{dt} + q \tag{7}$$

If y is twice continuously differentiable, $y \in C^2(I)$, then

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t) \tag{8}$$

Remark: Function $L[y]$ is continuous on I, $L[y] \in C(I)$

In terms of L,

- Nonhomogeneous equation $y'' + py' + qy = g$ is written as $\boxed{L[y] = g}$
- Homogeneous equation $y'' + py' + qy = 0$ is written as $\boxed{L[y] = 0}$

Important property of L: **L is a linear operator**

Linear Operators and Principle of Superposition

Linearity of L has an important consequence for homogeneous equations:

Principle of Superposition

If y_1 and y_2 are two solutions of

$$L[y] = y'' + py' + qy = 0$$

then any linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution.

Linear Operators and Principle of Superposition

All above results can be extended to the homogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{9}$$

We can define operator $\mathbf{K} : \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \mathbf{K}[\mathbf{x}]$

$$\mathbf{K}[\mathbf{x}] = \mathbf{x}' - \mathbf{P}(t)\mathbf{x} \tag{10}$$

Then, it is easy to show that

- $\mathbf{K}$ is a linear operator ($\mathbf{K}[c_1\mathbf{x}_1 + c_2\mathbf{x}_2] = c_1\mathbf{K}[\mathbf{x}_1] + c_2\mathbf{K}[\mathbf{x}_2]$)
- Principle of Superposition: If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two solutions of

  $$\mathbf{K}[\mathbf{x}] = \mathbf{x}' - \mathbf{P}(t)\mathbf{x} = 0$$

  then any linear combination

  $$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

  is also a solution.

Summary

- Existence and Uniqueness:
  - Let $p(t), q(t)$, and $g(t)$ be continuous functions on an open interval I
  - Let t_0 be any point in I, and
  - Let y_0 and y_1 be any given numbers.

  Then, on I, there exists a unique solution of the initial value problem

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_0, \\ y'(t_0) = y_1. \end{cases}$$

- Principle of Superposition for Homogeneous Equations:
  If y_1 and y_2 are two solutions of

$$y'' + p(t)y' + q(t)y = 0$$

  then any linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

  is also a solution.

Homework

<u>Homework:</u>

- Section 4.2
  - 1, 3, 5