Math 245 - Mathematics of Physics and Engineering I

Lecture 15. Second Order Linear ODEs: Definitions and Examples

February 15, 2012

Agenda

- Basic Definitions
  - 2nd order ODEs
  - Solutions
  - Initial Value Problems
  - Linear Equations
- Examples
  - The Spring-Mass System
  - The Linearized Pendulum
- Summary and Homework

Basic Definitions

Definition

A **second order ODE** is an equation involving the independent variable t, dependent variable $y = y(t)$ (unknown function), and its first and second derivatives, y' and y''.

$$F(t, y, y', y'') = 0 \tag{1}$$

We will always assume that it is possible to solve (1) for y'':

$$\boxed{y'' = f(t, y, y')} \tag{2}$$

Definition

A **solution** of (2) on an interval $I = (t_1, t_2)$ is a function $y = \phi(t)$ such that

- $\phi(t)$ is twice continuously differentiable on I, $\phi \in C^2(t_0, t_1)$
- $\phi(t)$ satisfies (2), $\phi(t)'' = f(t, \phi(t), \phi(t)')$

Basic Definitions

Definition

An **initial value problem** for a second order ODE is

$$\begin{cases} y'' = f(t, y, y'), \\ y(t_0) = y_0, \\ y'(t_0) = y_1. \end{cases} \tag{3}$$

where $t_0 \in I$, and y_0 and y_1 are given numbers.

Remark:

It is reasonable to expect that, to define a solution of a second order ODE uniquely, two initial conditions are needed, since two integrations are required to find a solution, and each integration introduces an arbitrary constant.

Basic Definitions

By introducing the state variables

$$x_1 = y, \qquad x_2 = y' \tag{4}$$

we can convert the second order ODE $y'' = f(t, y, y')$ to a system of two first order ODES:

$$\begin{cases} x_1' = x_2, \\ x_2' = f(t, x_1, x_2) \end{cases} \tag{5}$$

Sometimes, (5) is referred to as dynamical system. The evolution of the system state $\mathbf{x} = (x_1, x_2)^T$ in time is a trajectory or orbit in the phase plane x_1x_2.

Definition

The ODE $y'' = f(t, y, y')$ is said to be **linear** if it can be written in the **standard form**:

$$y'' + p(t)y' + q(t)y = g(t) \tag{6}$$

- if $g(t) = 0$, then the equation (6) is said to be **homogeneous**
- if $g(t) \neq 0$, then the equation (6) is said to be **nonhomogeneous**

The Spring-Mass System

- Forced, damped oscillator:

$$my''(t) + \gamma y'(t) + ky(t) = F(t)$$

- Forced, undamped oscillator:

$$my''(t) + ky(t) = F(t)$$

- Unforced, damped oscillator:

$$my''(t) + \gamma y'(t) + ky(t) = 0$$

- Unforced, undamped oscillator:

$$my''(t) + ky(t) = 0$$

The Linearized Pendulum

- Nonlinear equation:

$$\theta'' + \frac{\gamma}{mL}\theta' + \frac{g}{L}\sin\theta = 0$$

- If $\theta \approx 0$, then $\sin\theta \approx \theta$, and we obtain a linear equation:

$$\theta'' + \frac{\gamma}{mL}\theta' + \frac{g}{L}\theta = 0$$

Homework:

- Section 4.1
  - 1, 2, 3, 4, 5, 18