
February 8, 2012
Repeated Eigenvalues

We study homogeneous autonomous system:

$$\frac{dx}{dt} = Ax$$

In Lecture 10 and 11, we learn how to solve this system when eigenvalues $\lambda_1$ and $\lambda_2$ of matrix $A$ are real and different and complex conjugate, respectively.

The last (third) possibility for $\lambda_1$ and $\lambda_2$ is to be real and equal

$$\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$$

In this case there are two different possibilities for the corresponding eigenvectors:

- $v_1$ and $v_2$ are linearly independent, i.e. $\lambda$ has two independent eigenvectors.
- $v_1$ and $v_2$ are linearly dependent, i.e. $\lambda$ has only one independent eigenvector.
\[ \lambda \text{ has 2 independent eigenvectors} \]

It is easy to show that \( A \) has a repeated eigenvalue \( \lambda \) and two independent eigenvectors if and only if

\[
A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}
\]

In this case the general solution of

\[
\frac{dx}{dt} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} x
\]

is given by

\[
x = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix}
\]
\( \lambda \) has only one independent eigenvector

This is a more common case: matrix \( \mathbf{A} \) is nondiagonal. Let \( \mathbf{v} \) be the only independent eigenvector that corresponds to \( \lambda \). Then

\[
\mathbf{x}_1 = e^{\lambda t} \mathbf{v}
\]  

(1)

is a solution of the system \( d\mathbf{x}/dt = \mathbf{A}\mathbf{x} \). To find a fundamental set of solutions, we must find an additional solution. Let us look for another solution in the following form:

\[
\mathbf{x}_2 = t e^{\lambda t} \mathbf{v} + e^{\lambda t} \mathbf{w},
\]  

(2)

where \( \mathbf{w} \) is a vector to be determined. If (2) is a solution of the system, then \( \mathbf{w} \) must satisfy

\[
(\mathbf{A} - \lambda \mathbf{I}) \mathbf{w} = \mathbf{v}
\]  

(3)

**Definition**

The vector \( \mathbf{w} \) is called a **generalized eigenvector** corresponding to the eigenvalue \( \lambda \).

- Linear algebra: (3) can be always solved for \( \mathbf{w} \)
- Wronskian \( W[\mathbf{x}_1, \mathbf{x}_2] \neq 0 \Rightarrow \mathbf{x}_1 \) and \( \mathbf{x}_2 \) form a fundamental set
- The general solution is then \( \mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 \)
Examples

Find the general solution of the system

\[ \mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} \]

\[ \mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ 0 & -1/2 \end{pmatrix} \mathbf{x} \]
We study homogeneous autonomous system:

\[
\frac{dx}{dt} = Ax
\]

with repeated eigenvalues \( \lambda_1 = \lambda_2 = \lambda \).

- If \( A \) is diagonal, \( A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \), then the general solution is given by

\[
x = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix}
\]

- If \( A \) is nondiagonal, then a fundamental set of solution is formed by

\[
\begin{align*}
x_1 &= e^{\lambda t} v \\
x_2 &= te^{\lambda t} v + e^{\lambda t} w
\end{align*}
\]

where

\* \( v \) is the only independent eigenvector corresponding to \( \lambda \)

\* \( w \) is the generalized eigenvector corresponding to \( \lambda \), \( (A - \lambda I)w = v \)
Homework

Section 3.5

- Find the general solution: 3, 5
- Find the solution of the initial value problem: 9, 11