Math 245 - Mathematics of Physics and Engineering I

Lecture 12. Matrices, Determinants, Complex Variables: a Brief Overview

February 6, 2012

Agenda

- Linear Algebra
  - Matrices and operations with them
  - Determinants and their properties
- Complex Variables
  - Geometric Representation
  - Modulus, Argument
  - Euler's formula
- Homework

Matrices

Matrix is a fundamental object in mathematics

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

Elements a_{ij} may be real or complex numbers.

Definition

A matrix is an array of mathematical objects arranged in n rows and m columns.

- If $m = n$, then $\mathbf{A}$ is a square matrix.
- If $m = 1$, then $\mathbf{A}$ is a vector

Frequently the condensed notation is used:

$$\mathbf{A} = (a_{ij})$$

Basic Operations with Matrices

- Matrix Addition: The sum $\mathbf{A} + \mathbf{B}$ of two n-by-m matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is calculated componentwise:

$$\mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

- Product of a Matrix and a Scalar: The product of a matrix $\mathbf{A} = (a_{ij})$ and a scalar α is given by multiplying every entry of $\mathbf{A}$ by α:

$$\alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij})$$

- Transpose: The transpose of an n-by-m matrix $\mathbf{A}$ is the m-by-n matrix $\mathbf{A}^T$ formed by turning rows into columns and vice versa:

$$\mathbf{A}^T = (a_{ij})^T = (a_{ji})$$

Algebraic Properties

Theorem

If $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *are matrices and* α *and* β *are scalars, then*

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
- $\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A}$
- $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$
- $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$
- $(-1)\mathbf{A} = -\mathbf{A}$
- $-(-\mathbf{A}) = \mathbf{A}$
- $0\mathbf{A} = \mathbf{0}$
- $\alpha\mathbf{0} = \mathbf{0}$

All these properties simply extend the properties equivalent operations for real or complex numbers.

Matrix Multiplication

Definition

If $\mathbf{A} = (a_{ij})$ is an $n \times m$ matrix and $\mathbf{B} = (b_{ij})$ is an $m \times k$ matrix, then $\mathbf{AB}$ is defined to be the $n \times k$ matrix $C = (c_{ij})$ where

$$c_{ij} = \sum_{s=1}^{m} a_{is} b_{sj}, \qquad 1 \leq i \leq n, \;\; 1 \leq j \leq k$$

It is convenient to think of c_{ij} as the dot product of the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$

$$c_{ij} = (a_{i1}, \ldots, a_{im}) \begin{pmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{pmatrix} = \sum_{s=1}^{m} a_{is} b_{sj}$$

Remark: Note that the matrix product $\mathbf{AB}$ is defined only if the number of columns in the first factor $\mathbf{A}$ is equal to the number of rows in the second factor $\mathbf{B}$:

$$\underbrace{\mathbf{A}}_{n \times m} \; \underbrace{\mathbf{B}}_{m \times k} = \underbrace{\mathbf{C}}_{n \times k}$$

Properties of Matrix Multiplication

Theorem

Suppose that **A**, **B**, *and* **C** *are matrices for which the following products are defined and let* α *be a scalar. Then*

- $\boxed{\mathbf{AB} \neq \mathbf{BA}}$
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- $\mathbf{A}(\alpha\mathbf{B}) = (\alpha\mathbf{A})\mathbf{B} = \alpha(\mathbf{AB})$
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
- $\boxed{(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T}$

Inverse Matrix

Definition

The square $n \times n$ matrix $\mathbf{A}$ is said to be nonsingular or invertible it there is another matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

In this case, the matrix $\mathbf{B}$ is called the inverse of A and denoted as $\mathbf{B} = \mathbf{A}^{-1}$.

- $\mathbf{I}_n^{-1} = \mathbf{I}_n$
- If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular, then $\mathbf{AB}$ is nonsingular and $\boxed{(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}}$
- If $\mathbf{A}$ is nonsingular, then so is $\mathbf{A}^{-1}$, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- If $\mathbf{A}$ is nonsingular, then so is $\mathbf{A}^T$, and $\boxed{(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T}$

Determinants

With each square matrix $\mathbf{A}$, we can associate a number called its determinant, denoted by $\det \mathbf{A}$

$$\mathbf{A} \mapsto \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

- 1×1 matrix

$$\det(a_{11}) = a_{11}$$

- 2×2 matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Geometric interpretation: the absolute value of $\det \mathbf{A}$ equals to the area of the parallelogram defined by vectors $(a_{11}, a_{21})^T$ and $(a_{12}, a_{22})^T$.

Determinants

Determinants of square matrices of higher order are defined recursively.

If $\mathbf{A}$ is $n \times n$ matrix, denote by M_{ij}, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column from $\mathbf{A}$. M_{ij} is called the minor of a_{ij}. Then the expansion of $\det \mathbf{A}$ along the i^{th} row is defined by

$$\boxed{\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}} \qquad (1)$$

The expansion of $\det \mathbf{A}$ along the j^{th} column is defined by

$$\boxed{\det \mathbf{A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}} \qquad (2)$$

Remark: To compute $\det \mathbf{A}$, we can use either (1) or (2) for any i and j.

Properties of Determinants

Theorem

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *matrices. Then*

- *If* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by adding a constant multiple of one row (or column) to another row (or column), then* $\det \mathbf{B} = \det \mathbf{A}$
- *If* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by interchanging two rows or two columns, then* $\det \mathbf{B} = -\det \mathbf{A}$
- *If* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by multiplying any row or any column by a scalar* α*, then* $\det \mathbf{B} = \alpha \det \mathbf{A}$
- *If* $\mathbf{A}$ *has zero row or zero column, then* $\det \mathbf{A} = 0$
- *If* $\mathbf{A}$ *has two identical rows (or two identical columns), then* $\det \mathbf{A} = 0$
- *If one row (or column) of* $\mathbf{A}$ *is a constant multiple of another row (or column), then* $\det \mathbf{A} = 0$
- $\boxed{\det \mathbf{A} = \det \mathbf{A}^T}$
- $\boxed{\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}}$
- *If* $\mathbf{A}$ *is nonsingular, then* $\boxed{\det \mathbf{A}^{-1} = 1/\det \mathbf{A}}$

Complex Variables

- A complex number is $z = x + iy$, where x and y are real numbers and i satisfies $i^2 = -1$.
  - $x = \mathrm{Re}z$ is called the real part of z
  - $y = \mathrm{Im}z$ is called the imaginary part of z
- Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.
  - Geometric interpretation: it is convenient to think of $z = x + iy$ as about a vector with coordinates $(x, y)^T$.
  - In particular, $z = 0$ if and only if $\mathrm{Re}z = 0$ and $\mathrm{Im}z = 0$
- The complex conjugate of $z = x + iy$ is the complex number $\bar{z} = x - iy$.

$$\mathrm{Re}z = \frac{z + \bar{z}}{2} \qquad \mathrm{Im}z = \frac{z - \bar{z}}{2i}$$

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Complex Variables

- The absolute value or modulus of $z = x + iy$ is the nonnegative real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

- In geometric representation, the absolute value $r = |z| = |x + iy|$ is simply the length of the vector $\overrightarrow{OP} = (x, y)^T$. The angle θ between the positive real axis and vector $\overrightarrow{OP}$ is called the argument of $x + iy$, denoted $\arg(x + iy)$.

$$x = r\cos\theta \quad y = r\sin\theta$$

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

The right hand side is called the polar coordinate representation of $z = x + iy$.

- Euler's formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

  - Special case: if $\theta = \pi$, then $e^{i\pi} + 1 = 0$

Homework

Homework:

- Appendix A.1
  - 1
- Appendix A.3
  - Compute the determinant 6, 7
- Appendix B
  - 12, 21, 27