Math 245 - Mathematics of Physics and Engineering I

Lecture 11. Homogeneous Autonomous Systems: Complex Eigenvalues

February 3, 2012

Complex Eigenvalues

We study homogeneous autonomous system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{1}$$

In Lecture 10, we obtained the following result: if eigenvalues λ_1 and λ_2 of matrix $\mathbf{A}$ are real and different, then the general solution of (1) is given by

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \tag{2}$$

- $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $\mathbf{A}$ that correspond to λ_1 and λ_2, respectively
- c_1 and c_2 are arbitrary constants.

The other two possibilities for λ_1 and λ_2:

- λ_1 and λ_2 are complex conjugate
- λ_1 and λ_2 are real and equal

Complex Eigenvalues

Suppose that the eigenvalues of **A** are

$$\lambda_1 = \alpha + i\beta, \qquad \lambda_2 = \bar{\lambda}_1 = \alpha - i\beta, \qquad\qquad (\beta \neq 0) \tag{3}$$

Suppose also that $\mathbf{v}_1$ is an eigenvector corresponding to λ_1, then

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = 0 \tag{4}$$

By taking the conjugate of Eq. (4), we obtain

$$(\mathbf{A} - \bar{\lambda}_1 \mathbf{I})\bar{\mathbf{v}}_1 = 0 \tag{5}$$

Therefore, $\bar{\mathbf{v}}_1$ is an eigenvector corresponding to $\lambda_2 = \bar{\lambda}_1$. Thus,

Proposition

For a pair of complex conjugate eigenvalues, we can always choose the eigenvalues so that they are also complex conjugate.

Complex Eigenvalues

Using these eigenvalues and eigenvectors, we obtain two solutions of the system:

$$\mathbf{x}_1 = e^{(\alpha+i\beta)t}\mathbf{v}_1, \qquad \mathbf{x}_2 = e^{(\alpha-i\beta)t}\bar{\mathbf{v}}_1 \tag{6}$$

Question: Are we happy with solutions (6)?

Answer: No. Because we would like to have real-valued solutions. The coefficient matrix $\mathbf{A}$ is real-valued, so it is natural to look for real-valued solutions.

Our goal: to find a fundamental set of real-valued solutions

Recall that, by Euler's formula,

$$e^{i\beta t} = \cos(\beta t) + i\sin(\beta t) \tag{7}$$

and let $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real-valued vectors. Then from (6) and (6), we obtain

$$\mathbf{x}_1 = \mathbf{u} + i\mathbf{w}, \qquad \mathbf{x}_2 = \mathbf{u} - i\mathbf{w} \tag{8}$$

where

$$\mathbf{u} = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t), \qquad \mathbf{w} = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t) \tag{9}$$

Complex Eigenvalues

Theorem

- *Both* $\mathbf{u} = e^{\alpha t}(\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t)$ *and* $\mathbf{w} = e^{\alpha t}(\mathbf{a} \sin \beta t + \mathbf{b} \cos \beta t)$ *are solutions of homogeneous autonomous system*

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

- *The Wronskian of* $\mathbf{u}$ *and* $\mathbf{w}$ *is not nonzero.*

$$W[\mathbf{u}, \mathbf{w}](t) \neq 0$$

  Therefore, $\mathbf{u}$ *and* $\mathbf{w}$ *form a fundamental set of solutions.*

- *The general solution of the system is can be written as*

$$\mathbf{x} = c_1\mathbf{u} + x_2\mathbf{w}$$

Example

- Find a fundamental set of solutions of the following system

$$\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{x}$$

Summary

- If the eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = \alpha + i\beta, \qquad \lambda_2 = \bar{\lambda}_1 = \alpha - i\beta, \qquad\qquad (\beta \neq 0) \tag{10}$$

with corresponding eigenvectors

$$\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}, \qquad \mathbf{v}_2 = \bar{\mathbf{v}}_1 = \mathbf{a} - i\mathbf{b}$$

then a fundamental set of real-valued solutions of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

is given by

$$\boxed{\begin{aligned} \mathbf{x}_1 &= \mathrm{Re}(e^{\lambda_1 t}\mathbf{v}_1) = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) \\ \mathbf{x}_2 &= \mathrm{Im}(e^{\lambda_1 t}\mathbf{v}_1) = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t) \end{aligned}}$$

and the general solution is

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

Homework

Homework:

- Section 3.4
  - Find the general solution: 3, 5
  - Find the solution of the initial value problem: 7, 9