Math 245 - Mathematics of Physics and Engineering I

Lecture 10. Homogeneous Autonomous Systems

February 1, 2012

Agenda

- Homogeneous Systems and their importance
- The Eigenvalue Method for Solving $\mathbf{x}' = \mathbf{A}\mathbf{x}$
- Real and Different Eigenvalues
- Wronskian
- Fundamental Set of Solutions
- Summary and Homework

Homogeneous Systems

An autonomous system of two first order linear ODEs has the following form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b} \qquad (1)$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{b}$ is a constant vector.

Definition

System (1) is called **homogeneous** if $\mathbf{b} = 0$.

Important message: if $\mathbf{A}$ is nonsingular, then it is possible to reduce a nonhomogeneous system to a homogeneous one

If $\mathbf{A}$ is nonsingular, then (1) has a unique equilibrium solution:

$$\mathbf{x}_{\mathrm{eq}} = -\mathbf{A}^{-1}\mathbf{b}$$

Let

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_{\mathrm{eq}}$$

Then

$$\frac{d\tilde{\mathbf{x}}}{dt} = \mathbf{A}\tilde{\mathbf{x}} \qquad (2)$$

Homogeneous Systems

Thus, if $\tilde{\mathbf{x}}$ is a solution of the homogeneous system

$$\frac{d\tilde{\mathbf{x}}}{dt} = \mathbf{A}\tilde{\mathbf{x}}$$

then the solution of the nonhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

is given by

$$\boxed{\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{x}_{\mathrm{eq}} = \tilde{\mathbf{x}} - \mathbf{A}^{-1}\mathbf{b}}$$

Thus, to solve a nonhomogeneous autonomous system (with nonsingular $\mathbf{A}$), we need

- Find its equilibrium solution (linear algebra problem)
- Solve the corresponding homogeneous system

This shows that homogeneous systems are of fundamental importance

The Eigenvalue Method for Solving $x' = Ax$

Our goal: to learn how to solve homogeneous autonomous systems

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{3}$$

Let us look for solutions of (3) in the following form:

$$\mathbf{x} = e^{\lambda t}\mathbf{v} \tag{4}$$

where $\mathbf{v}$ is a constant vector and λ is a scalar to be determined.

Remark: Why? Motivation comes from a simple case $\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$

It follows from (3) and (4) that

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0 \tag{5}$$

Thus, $\mathbf{x} = e^{\lambda t}\mathbf{v}$ is a solution of (3), if

- λ is an eigenvalue of $\mathbf{A}$
- $\mathbf{v}$ is a corresponding eigenvector of $\mathbf{A}$

Example

- Find at least one nontrivial solution of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

The Eigenvalue Method for Solving $x' = Ax$

The eigenvalues λ_1 and λ_2 are the roots of the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{vmatrix} = \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0 \tag{6}$$

For each eigenvalue λ, we can solve

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0$$

and obtain corresponding eigenvector $\mathbf{v}$.

In general, there three possibilities for λ_1 and λ_2:

- λ_1 and λ_2 are real and different
- λ_1 and λ_2 are real and equal
- λ_1 and λ_2 are complex conjugate

Real and Different Eigenvalues

Assume that λ_1 and λ_2 are real and different. Then, using the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, we can write down two solutions of

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

Namely,

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \qquad \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2$$

Principle of Superposition

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are (any) solutions of

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

Then, any linear combination of these two solutions

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

is also a solution.

- $\Rightarrow$ $\boxed{\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2}$ is a solution for any constants c_1, c_2.

Wronskian

Suppose now that we also have an initial condition: $\mathbf{x}(t_0) = \mathbf{x}_0 = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$

Question: Is it possible to find c_1 and c_2 such that $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ satisfies (7)?
We need to solve for c_1 and c_2 the following equation:

$$c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) = \mathbf{x}_0 \tag{7}$$

In more detail:

$$\begin{pmatrix} x_1^1(t_0) & x_2^1(t_0) \\ x_1^2(t_0) & x_2^2(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} \tag{8}$$

System (8) has a unique solution $\Leftrightarrow \begin{vmatrix} x_1^1(t_0) & x_2^1(t_0) \\ x_1^2(t_0) & x_2^2(t_0) \end{vmatrix} \neq 0$

Definition

The determinant

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} x_1^1(t) & x_2^1(t) \\ x_1^2(t) & x_2^2(t) \end{vmatrix}$$

is called the Wronskian.

Fundamental Set of Solutions

If solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ are given by

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \qquad \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2 \tag{9}$$

then their Wronskian is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{vmatrix} e^{(\lambda_1+\lambda_2)t} = \begin{vmatrix} v_1^1 & v_2^1 \\ v_1^2 & v_2^2 \end{vmatrix} e^{(\lambda_1+\lambda_2)t}$$

- $e^{(\lambda_1+\lambda_2)t} \neq 0$
- Since $\lambda_1 \neq \lambda_2$, $\begin{vmatrix} v_1^1 & v_2^1 \\ v_1^2 & v_2^2 \end{vmatrix} \neq 0$ (see Lecture 8)

Therefore, the Wronskian of the vectors (9) is nonzero.

Definition

Two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ whose Wronskian is not zero are referred to as a **fundamental set of solutions**.

$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2$ is a fundamental set of solutions of $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$

Importance of a Fundamental Set of Solutions

Theorem

Suppose that $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are solutions of*

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{10}$$

and suppose that their Wronskian is not zero

$$W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$$

Then

1. $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *form a fundamental set of solutions*
2. *the general solution of (10) is given by*

   $$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

   where c_1 *and* c_2 *are arbitrary constants.*
3. *if there is a given initial condition* $\mathbf{x}(t_0) = \mathbf{x}_0$*, then this condition determines the constants* c_1 *and* c_2 *uniquely.*

Example

- Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

- Find the solution that also satisfies the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Summary

- Homogeneous autonomous systems $d\tilde{\mathbf{x}}/dt = \mathbf{A}\tilde{\mathbf{x}}$ are of fundamental importance since (if $\mathbf{A}$ is nonsingular) it is possible to reduce a nonhomogeneous system $d\mathbf{x}/dt = \mathbf{A}\mathbf{x} + \mathbf{b}$ to a homogeneous one

$$\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{x}_{\text{eq}} = \tilde{\mathbf{x}} - \mathbf{A}^{-1}\mathbf{b}$$

- If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$, then the determinant

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} x_1^1(t) & x_2^1(t) \\ x_1^2(t) & x_2^2(t) \end{vmatrix}$$

  is called the Wronskian
- $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions $\Leftrightarrow W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$
- How to solve $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$? The Eigenvalue Method:
  - Let $\lambda_1 \neq \lambda_2$ be two different real eigenvalues of $\mathbf{A}$, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the corresponding eigenvectors.
  - Then the general solution of $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$ is

$$\boxed{\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2}$$

Homework

Homework:

- Section 3.3
  - find the general solution: 3, 11
  - solve the initial value problem: 14, 16